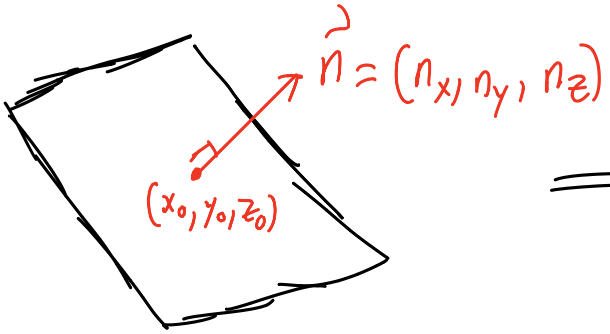
Vol 1.

(Ch1) Lines and planes



$\Rightarrow$ Eqn plane:

$$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0$$

$\Rightarrow$ The set of all points (x,y,z) satisfying this equation lay on a plane.

(ch2) curves and surfaces

To parametrize a line in, say, 3d space:

- Idea: A line in any dimension of space is defined by two points.

$\Rightarrow$ Between these two points, we have a $\Delta x, \Delta y, \Delta z$.

parametrization of a line in the direction of $(\Delta x, \Delta y, \Delta z)$ passing through (x_0, y_0, z_0)

$$\begin{cases} x(t) = \Delta x\, t + x_0 \\ y(t) = \Delta y\, t + y_0 \\ z(t) = \Delta z\, t + z_0 \end{cases} \Longrightarrow \left(\vec{r}(t) = \vec{r}_0 + t\vec{v}\right)$$

$t \geq 0$, $t \leq 0$, $-\infty < t < \infty$, it won't matter for now.

Ex Write the parametrization for the line parallel to the segment joining the points $(3,1,2)$, $(2,-1,0)$ offset to include $(0,1,-1)$.

$\Delta x = 2-3=-1$, $x_0 = 0$

$\Delta y = -1-1=-2$, $y_0 = 1$

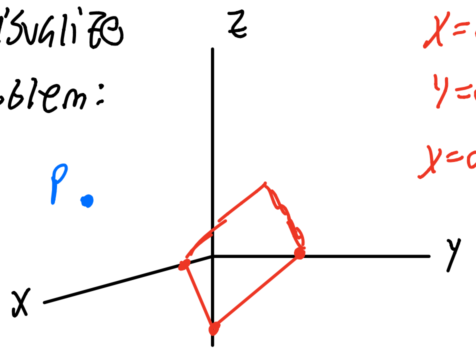
$\Delta z = 0-2=-2$, $z_0 = -1$

$$\boxed{\begin{cases} x(t) = -t \\ y(t) = -2t+1 \\ z(t) = -2t-1 \end{cases}}$$

at $t=0$: $(0,1,-1)$

Ex Write a parametrization of the line passing through $P(5,-1,4)$ and parallel to the plane $4x+3y-z=3$.

Let's visualize the problem:



$x=0, z=0, \ y=1$

$y=0, \ z=0, \ x=\frac{3}{4}$

$x=0, y=0, \ z=-3$

$\Longrightarrow$ Is $(5,-1,4)$ on the plane?

$4(5)+3(-1)-4 \overset{?}{=} 3$

$13 \neq 3$. No, P is Not in the plane.

So what can we do?

$\Longrightarrow$ Find an equation of the appropriate line in the plane, then offset it to our point P!

Remember: A line is defined by two points.

→ Do we have two points in the plane?

Yes! We have 3! (intercepts)

Let's use these → $(0,1,0)$

→ $(0,0,-3)$

$(\frac{3}{4},0,0)$

$\Delta x=0, \ \Delta y=1, \ \Delta z=3$

$$\boxed{\begin{aligned} x(t)&=0+5=5\\ y(t)&=t-1\\ z(t)&=3t+4 \end{aligned}}$$

Angle btwn line and normal to plane?

90°

Ex Planes $x+y+z=-1$ and $x+2y+3z=-4$ intersect in a line. Parametrize that line.

$\Longrightarrow$ We need an eqn for all (x,y,z) that satisfy <u>both</u> plane equations.

$$\Longrightarrow \begin{array}{r} x+y+z=-1 \\ -(x+2y+3z=-4) \\ \hline -y-2z=3 \end{array}$$

$$y=-2z-3$$

This is the collection of all z's and y's satisfying both plane equations

Let:

$z(t)=t$

$y(t)=-2t-3$

$\Longrightarrow$ Find $x(t)$ using $y(t)$, $z(t)$:

$$x-2t-3+t=-1$$
$$x-t=2$$
$$\underline{x=t+2}$$

$$\Longrightarrow \boxed{\begin{cases} x(t)=t+2 \\ y(t)=-2t-3 \\ z(t)=t \end{cases}}$$

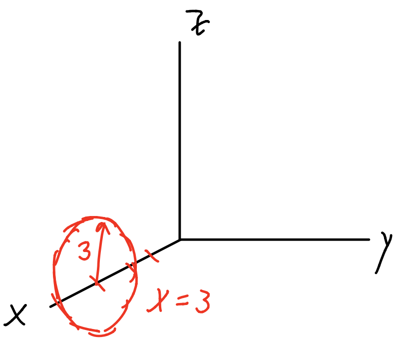
Another thing that can happen: given parametric equations, find the implicit form.

Ex Find the implicit form:

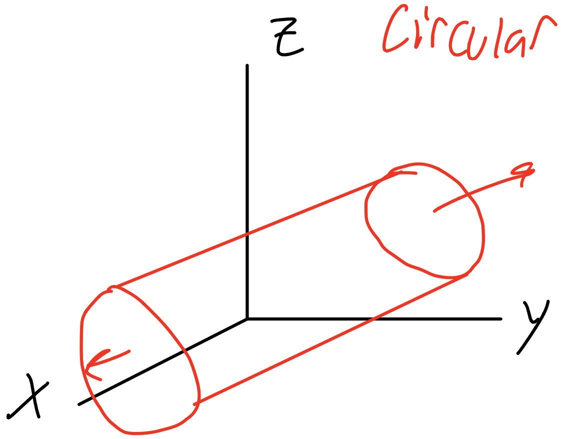
$$x(t) = 3$$
$$y(t) = 3\cos t$$
$$z(t) = 3\sin t$$

$$\begin{cases} y^2+z^2 = 9\cos^2 t + 9\sin^2 t = 9 \\ x = 3 \end{cases}$$

Sketch:



Consider: What if we had $y(t) = 3\cos t$, $z(t) = 3\sin t$, x unspecified? Cylinder!



Circular cross sections in yz plane.

Sketching Surfaces in 3D:

1) Make equation look nice/familiar.
2) Set $x=0$, $y=0$, $z=0$ and sketch in "cross sections"
3) Adjust values of x, y, z to form a picture.

Ex $y-4x^2 = 2z^2$

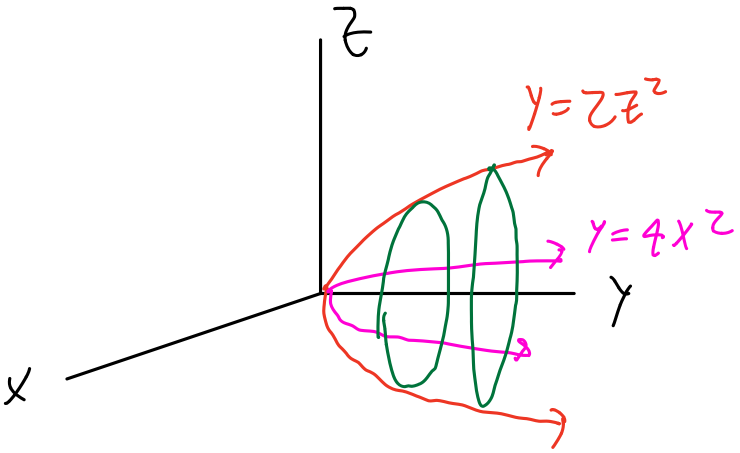
1) Make equation look nice:

$\Rightarrow y = 4x^2 + 2z^2$

2) $x=0$, $y=2z^2$ Parabola in yz plane	$z=0$, $y=4x^2$ Parabola in xy plane

$y=0$, $4x^2+2z^2=0 \Longrightarrow x=z=0=y$ (origin)

Also note y can never be negative:



We have a paraboloid:

$$y = \frac{x^2}{1/4} + \frac{z^2}{1/2}$$

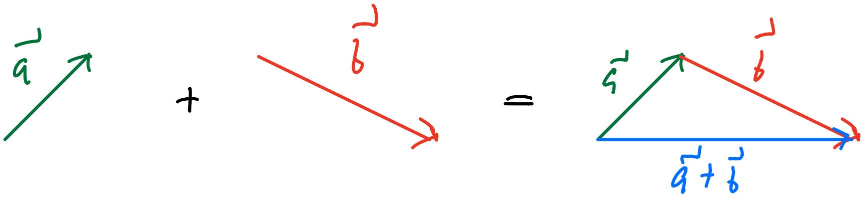
(Ch4) Vectors

Unit Vectors: magnitude 1: $\|\vec{u}\| = 1$

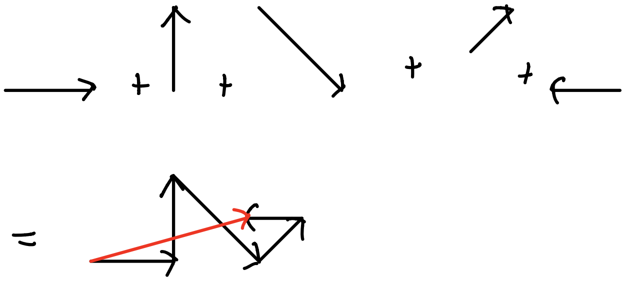
$$\Longrightarrow \vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\Longrightarrow \vec{u} = \begin{bmatrix} \sqrt{2}/2 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \text{ (example)}$$

Vector addition



- Put tip of $\vec{a}$ to tail of $\vec{b}$
- Resultant: Draw vector from tail of first vector $(\vec{a})$ to tip of final vector $(\vec{b})$.

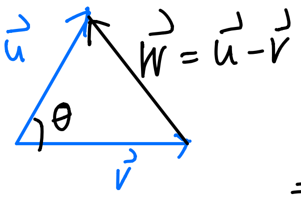


(Ch5) Dot Products

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \|\vec{u}\| \|\vec{v}\| \cos\theta \Longrightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \ \|\vec{v}\|}$$

Measures how parallel two vectors are.

Optional: Where it comes from:



From law of cosines:

$$|\vec{W}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

$$2|\vec{u}||\vec{v}|\cos\theta = |\vec{u}|^2 + |\vec{v}|^2 - |W|^2$$

$$\Longrightarrow \vec{W} = \vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$$

$$|\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2, \quad |\vec{v}|^2 = v_1^2 + v_2^2 + v_3^2$$

$$\Longrightarrow |\vec{W}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$

$$\Rightarrow |\vec{w}|^2 = u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^3 - 2u_3v_3 + v_3^2$$

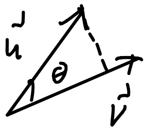
$$\Rightarrow |\vec{u}|^2 + |\vec{v}|^2 - |\vec{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3)$$

$$\Rightarrow 2|\vec{u}||\vec{v}|\cos\theta = |\vec{u}|^2 + |\vec{v}|^2 - |\vec{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3)$$

$$\underline{|\vec{u}||\vec{v}|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3 \equiv \vec{u} \bullet \vec{v}}$$

Projections

Ex | If $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, What is the projection of $\vec{u}$ in the direction of $\vec{v}$?



$$\text{Proj}(\vec{u} \text{ in dir of } \vec{v}) = \|\vec{u}\|\cos\theta = \frac{\|\vec{u}\|\,\|\vec{v}\|\cos\theta}{\|\vec{v}\|} = \frac{\vec{u} \bullet \vec{v}}{\|\vec{v}\|} = \vec{u} \bullet \frac{\vec{v}}{\|\vec{v}\|}$$

$$\|\vec{v}\| = \sqrt{9+16} = \sqrt{25} = 5$$

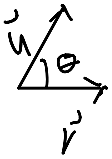
$$\vec{u} \bullet \vec{v} = (1)(3) + (2)(0) + (1)(4) = 7$$

$$\Rightarrow \boxed{7/5}$$

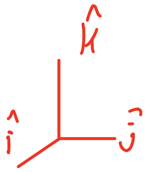
$$\text{Proj}(\vec{v} \text{ in dir of } \vec{u}) = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|} = \frac{7}{\sqrt{1+4+1}} = \boxed{\frac{7}{\sqrt{6}}}$$

(Ch 6) Cross Products

- Cross products: Measure how perpendicular two vectors are



$\vec{u} \times \vec{v}$ into page
$\vec{v} \times \vec{u}$ out of page $\}$ $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\,\|\vec{v}\| \sin\theta$



Ex | $3\hat{i} \times (1\hat{i} + 4\hat{j} - 1\hat{k}) = (3)(1)(\hat{i} \times \hat{i}) + (3)(4)(\hat{i} \times \hat{j}) + (3)(-1)(\hat{i} \times \hat{k})$

$$= 0 + 12\hat{k} - 3(-\hat{j}) = \boxed{3\hat{j} + 12\hat{k}}$$

- Cross product $\vec{u}\times\vec{v}$ gives vector normal to $\vec{u}$ and $\vec{v}$ (by def, $\hat{i}\times\hat{j}=\hat{k}$).

Ex Find the equation of the plane spanned by vectors

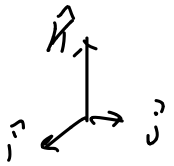
$\vec{u}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$, $\vec{v}=\begin{bmatrix}1\\1\\0\end{bmatrix}$ and containing the point $(1,2,2)$.

Plane eqn: $n_x(x-x_0)+n_y(y-y_0)+n_z(z-z_0)=0$

$(x_0,y_0,z_0)=(1,2,2)$ ✓

Find $\vec{n}=(n_x,n_y,n_z)$

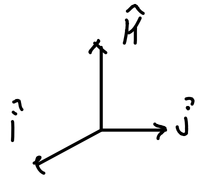
$\vec{n}=\vec{u}\times\vec{v}$. Could also use $\vec{v}\times\vec{u}$ $(=-\vec{u}\times\vec{v})$.



$$\vec{u}\times\vec{v}=(1\hat{i}+0\hat{j}-1\hat{k})\times(1\hat{i}+1\hat{j}+0\hat{k})$$
$$=(1)(1)(\hat{i}\times\hat{i})+(1)(1)(\hat{i}\times\hat{j})+(-1)(1)(\hat{k}\times\hat{i})+(-1)(1)(\hat{k}\times\hat{j})$$
$$=\hat{k}-\hat{j}+\hat{i}$$
$$\vec{n}=\hat{i}-\hat{j}+\hat{k}\ ;\ n_x=1,\ n_y=-1,\ n_z=1$$
$$\Longrightarrow \boxed{1(x-1)-1(y-2)+1(z-2)=0}$$

Ex Parametrize the intersection of the planes with normal vectors

$\vec{n}_1=\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{n}_2=\begin{bmatrix}2\\1\\1\end{bmatrix}$ if the point $(3,1,4)$ is on this line.



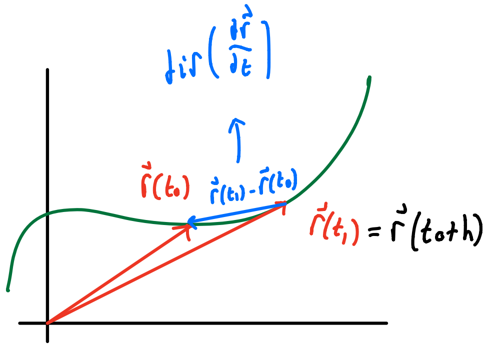
normal to plane: $\vec{n}=\vec{n}_1\times\vec{n}_2$ (or $\vec{n}_2\times\vec{n}_1$)
$$=(\hat{i}+\hat{j})\times(2\hat{i}+\hat{j}+\hat{k})$$
$$=\hat{k}-\hat{j}-2\hat{k}+\hat{i}$$
$$=\hat{i}-\hat{j}-\hat{k}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$$
$$\Longrightarrow \boxed{(x-3)-(y-1)-(z-4)=0}$$

(Ch 7) Intro to Vector Calculus

$\vec{r}(t)$ is the position vector, anchored at the origin.

Sometimes denoted $\vec{X}(t)$.

$$\frac{d\vec{r}}{dt} = \lim_{h\to 0}\frac{\vec{r}(t_0+h)-\vec{r}(t_0)}{h}$$



$$\Longrightarrow \vec{r}(t)=\begin{bmatrix}\cos(t)\\ \sin(3t)\\ t^3\end{bmatrix} \Longrightarrow \vec{r}'(t)=\begin{bmatrix}-\sin t\\ 3\cos 3t\\ 3t^2\end{bmatrix}$$

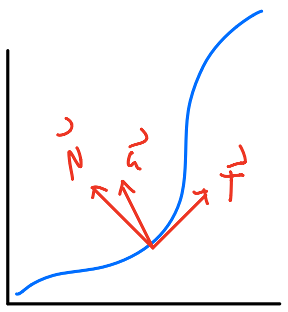
Treat each Component seperately

(Ch 8) Vector Calculus and Motion

$\vec{T},\vec{N},\vec{B}$ frame:

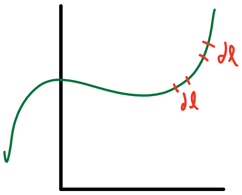
Here's the idea...

- For any point on a curve, there is a
  - tangent vector } orthogonal.
  - normal vector
  - acceleration vector.



- The $\vec{T},\vec{N},\vec{B}$ frame is useful when it is impossible to use a standard coordinate system to a problem, as in general relativity.

$$\Longrightarrow \vec{N}\cdot\vec{T}=0 \;\Big|\; \vec{T}=\frac{\vec{v}}{\|\vec{v}\|} \;\Big|\; \vec{N}=\frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$



∂l: Small arclength element

$\frac{\partial l}{\partial t} = \lim_{\Delta t \to 0} \frac{\Delta l}{\Delta t}$ = Speed!

$= \|\vec v\|$

"Givens":

$\vec T = \frac{\vec v}{\|\vec v\|}$, $\vec N = \frac{\vec T'}{\|\vec T'\|}$

$\frac{\partial l}{\partial t} = \|\vec v\|$

↑

$$\vec a = \frac{\partial \vec v}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\vec v}{\|\vec v\|}\frac{\partial l}{\partial t}\right) = \frac{\partial}{\partial t}\left(\vec T \frac{\partial l}{\partial t}\right) = \frac{\partial \vec T}{\partial t}\frac{\partial l}{\partial t} + \vec T \frac{\partial^2 l}{\partial t^2}$$ ← Product rule

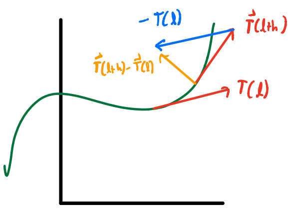
$$= \left(\frac{\partial \vec T}{\partial l}\frac{\partial l}{\partial t}\right)\frac{\partial l}{\partial t} + \vec T\frac{\partial^2 l}{\partial t^2}$$

$$= \frac{\partial \vec T}{\partial l}\left(\frac{\partial l}{\partial t}\right)^2 + \vec T \frac{\partial^2 l}{\partial t^2}$$

We have two $\frac{\partial l}{\partial t}$'s and a $\frac{\partial \vec T}{\partial l}$. How else can we express $\frac{\partial \vec T}{\partial l}$??

What is the direction of $\frac{\partial \vec T}{\partial l}$??

unit tangent as function of arclength



$$\frac{\partial \vec T(l)}{\partial l} = \lim_{h \to 0} \frac{\vec T(l+h) - \vec T(l)}{h}$$

$\vec T(l+h) - \vec T(l)$ is Normal to curve!

$$\therefore \quad \text{direction}\left(\frac{\partial \vec T(l)}{\partial l}\right) = \vec N$$

"Some multiple"
↓

So, $\frac{\partial \vec T}{\partial l}$ will be some multiple of $\vec N$: $\frac{\partial \vec T}{\partial l} = K\vec N$

This gives us intuition for K!!

If K is big, $\frac{\partial \vec T}{\partial l}$ is big, so the unit tangent vector is changing a lot. I.E. greater curvature!

$$\vec a = K\vec N\left(\frac{\partial l}{\partial t}\right)^2 + \frac{\partial^2 l}{\partial t^2}\vec T$$

$$= K\vec N\|\vec v\|^2 + \frac{\partial}{\partial t}\frac{\partial l}{\partial t}\vec T$$

$$= \boxed{K\|\vec v\|^2\vec N + \frac{\partial}{\partial t}\|\vec v\|\vec T}$$

Let $a_N = K\|\vec v\|^2$; $a_T = \frac{\partial}{\partial t}\|\vec v\|$

$$\therefore \boxed{\vec a = a_N\vec N + a_T\vec T}$$

a_N: how tightly you turn

a_T: How quickly you speed up or slow down

Resume Here:

Binormal: $\vec{B} = \vec{T} \times \vec{N}$

$\vec{T}$ and $\vec{N}$ are both unit.

$\therefore \vec{B}$ is unit!

$\boxed{\text{Ex}}$ Find $\vec{T}, \vec{N}, \vec{B}$ for $\vec{r}(t) = \cos(2t)\hat{i} + \sin(2t)\hat{j}$

$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$. $\vec{V} = -2\sin(2t)\hat{i} + 2\cos(2t)\hat{j}$ $\Longrightarrow \|\vec{v}\| = 2$

$$\Longrightarrow \boxed{\vec{T} = -\sin(2t)\hat{i} + \cos(2t)\hat{j}}$$

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|}.$$

$$\vec{T}' = -2\cos(2t)\hat{i} - 2\sin(2t)\hat{j}$$

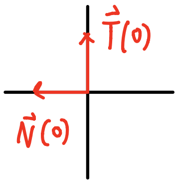
$$\|\vec{T}'\| = 2$$

$$\Longrightarrow \boxed{\vec{N} = -\cos(2t)\hat{i} - \sin(2t)\hat{j}}$$

Check: $\vec{T} \cdot \vec{N} = 0$ ✓

$\Longrightarrow \vec{B} = \vec{T} \times \vec{N} = \pm\hat{k}$ in this case

But which one? Choose a point: $\underline{t=0}$; $\vec{T} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{N} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



$$\boxed{B = +\hat{k}}$$

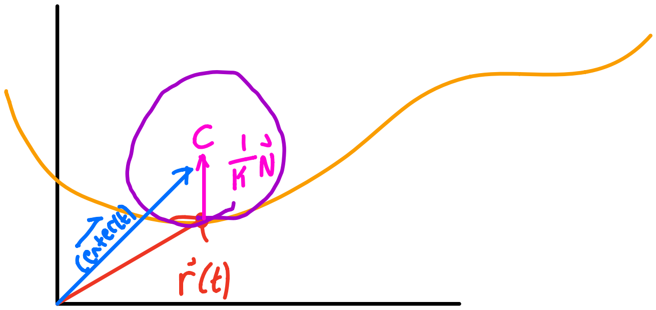
Curve stays in x-y plane (expected).

If time: Oscullating Circle : "Best fit Circle to Curve"

$\frac{1}{\kappa(t)}$ = radius of oscullating circle (Small curvature, big radius)

Hence, the Center of the Circle is given:

$$\overrightarrow{Center}(t) = \vec{r}(t) + \frac{1}{\kappa(t)}\vec{N}(t)$$



(Ch 9) Matrices : 3x3 example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\Longrightarrow$ Transpose: Switches rows w/ columns, Columns w/ rows.

Ex

$$A = \begin{bmatrix} 2 & 1 & 2 & 5 \\ 0 & 4 & 5 & 0 \\ 3 & 7 & 1 & 5 \end{bmatrix}$$

Size(A): 3x4

size(A^T) = 4x3

$$A^T = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 7 \\ 2 & 5 & 1 \\ 5 & 0 & 5 \end{bmatrix} \Longrightarrow (A^T)^T = A$$

(Ch 10) Matrix Multiplication

Recall if we have a vector $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\Rightarrow \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

$\Rightarrow$ Matrix multiplication is a way of taking a bunch of dot products: And it's defined that way!

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \overset{B}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}} = \overset{C}{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}$$

c_{11} = (1st row of A) • (1st column B) = first row, first col entry (C)

c_{12} = (1st row of A) • (2nd column B) = first row, second col entry (C)

c_{21} = (2nd row of A) • (1st column B) = Second row, first col entry (C)

c_{22} = (2nd row of A) • (2nd column B) = Second row, Second col entry (C)

This is all captured with a summation:

$$C_{ij} = (AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} = (i^{th} \text{ row of } A) \cdot (j^{th} \text{ col } B)$$

But you can't dot a 3d vector with a 2d vector, So this's dot product definition imposes some conditions.

$\Rightarrow$ For AB: If A is $m \times n$, B must be $n \times p$, AB is $m \times p$

Ex

$$\overset{2\times 3}{\begin{bmatrix}4&2&1\\3&1&2\end{bmatrix}}\overset{3\times 2}{\begin{bmatrix}1&4\\1&2\\2&1\end{bmatrix}} = \overset{2\times 2}{\begin{bmatrix}\underline{4(1)+2(1)+1(2)} & \underline{4(4)+2(2)+1(1)}\\ \underline{3(1)+1(1)+2(2)} & \underline{3(4)+1(2)+2(1)}\end{bmatrix}} = \overset{2\times 2}{\begin{bmatrix}8&21\\8&16\end{bmatrix}}$$

The most important thing: Remember this!!

$$\overset{3\times 3}{\begin{bmatrix}1&2&3\\2&1&1\\0&2&4\end{bmatrix}}\overset{3\times 1}{\begin{bmatrix}3\\2\\4\end{bmatrix}}.$$

Note, when multiplying with fingers, the 3 is always "selected" when pointing to the first entry of each row of the 3x3. This pattern continues.

$$\Rightarrow 3\begin{bmatrix}1\\2\\0\end{bmatrix}+2\begin{bmatrix}2\\1\\2\end{bmatrix}+4\begin{bmatrix}3\\1\\4\end{bmatrix} = \begin{bmatrix}3+4+12\\6+2+4\\0+4+16\end{bmatrix} = \begin{bmatrix}19\\12\\20\end{bmatrix}$$

$$= \begin{bmatrix}1&2&3\\2&1&1\\0&2&4\end{bmatrix}\begin{bmatrix}3\\2\\4\end{bmatrix}$$

Caution: $AB \neq BA$ generally, but $A(BC) = (AB)C$

You can verify this, but these properties will be given a beautiful geometric interpretation when we cover linear transformations.

(Ch11) Matrix Equations

⟹ Linear systems of equations can be encoded in matrices: $A\vec{x}=\vec{b}$

Ex

$$\begin{aligned} x-3y+3z&=8\\ 5x+y-2z&=7\\ -2x+y-z&=-6 \end{aligned} \qquad \left\{ \underset{A}{\begin{bmatrix} 1 & -3 & 3\\ 5 & 1 & -2\\ -2 & 1 & -1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x\\ y\\ z \end{bmatrix}} \underset{=}{=} \underset{\vec{b}}{\begin{bmatrix} 8\\ 7\\ -6 \end{bmatrix}} \right.$$

$$\Rightarrow \text{Soln: } \vec{x}=\begin{bmatrix} 2\\ -1\\ 1 \end{bmatrix}=\begin{bmatrix} x\\ y\\ z \end{bmatrix}$$

(Ch12) Row Reduction

Idea: given a system that looks like this:

$$\overset{3\times 3}{\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} x\\ y\\ z \end{bmatrix}} = \begin{bmatrix} 3\\ 2\\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 3\\ 2\\ 1 \end{bmatrix}$$

↑

This is a diagonal matrix: If we can make a system into a diagonal matrix, it is much easier to find a solution!

Row Operations: To our system, we can do the following...

$$\begin{aligned} x-3y+3z&=8\\ 5x+y-2z&=7\\ -2x+y-z&=-6 \end{aligned} \quad \overset{\text{Switch Eqns}}{\iff} \quad \begin{aligned} 5x+y-2z&=7\\ x-3y+3z&=8\\ -2x+y-z&=-6 \end{aligned} \quad \overset{\text{Rescale Equations}}{\iff} \quad \begin{aligned} 10x+2y-4z&=14\\ 3x-9y+9z&=24 \quad &\text{Eq2}\\ -8x+4y-4z&=-24 \quad &\text{Eq3} \end{aligned}$$

$$\overset{\text{Adding Eqns.}}{\iff} \quad \begin{aligned} 10x+2y-4z&=14\\ -5x-5y+5z&=0 \quad &(\text{Eq 2}+\text{Eq 3}) \end{aligned}$$

These operations all apply to the augmented matrix, which is a way to consolidate information:

$$\implies \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -3 & 3 & 8\\ 5 & 1 & -2 & 7\\ -2 & 1 & -1 & -6 \end{array}\right] \end{array} \underset{\text{Clear first Col}}{\overset{R_2: R_2-5R_1}{\sim}} (8\cdot 5=40) \left[\begin{array}{ccc|c} 1 & -3 & 3 & 8\\ 0 & 16 & -17 & -33\\ -2 & 1 & -1 & -6 \end{array}\right]$$

$$\overset{R_3: R_3+2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 8\\ 0 & 16 & -17 & -33\\ 0 & -5 & 5 & 10 \end{array}\right] \overset{R_3: \frac{1}{5}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 8\\ 0 & 16 & -17 & -33\\ 0 & -1 & 1 & 2 \end{array}\right] \overset{R_2: R_2+16R_3}{\sim} (16)(2)=32 \left[\begin{array}{ccc|c} 1 & -3 & 3 & 8\\ 0 & 0 & -1 & -1\\ 0 & -1 & 1 & 2 \end{array}\right]$$

Clearing 2nd col.

$$\overset{\substack{R_2: -R_2\\ R_3: -R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 8\\ 0 & 0 & 1 & 1\\ 0 & 1 & -1 & -2 \end{array}\right] \overset{\substack{R_1: R_1+3R_3\\ R_3: R_3+R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 1\\ 0 & 1 & 0 & -1 \end{array}\right] \overset{R_2 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1 \end{array}\right]$$

Clearing 3rd col.

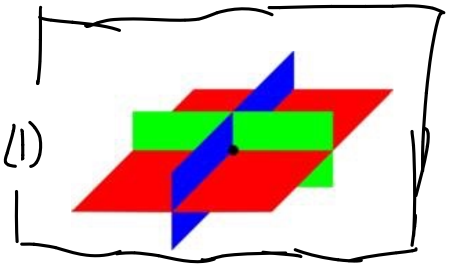
Soln!

$$\boxed{\begin{aligned} 1x&=2\\ 1y&=-1\\ 1z&=1 \end{aligned}}$$

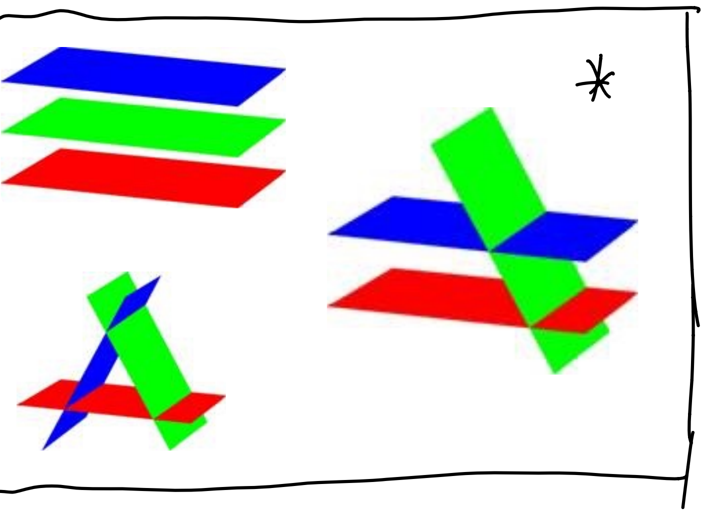
Bonus: The 3 different soln types we can get from a system:

1) Unique soln (already shown)
2) No soln
3) ∞ soln

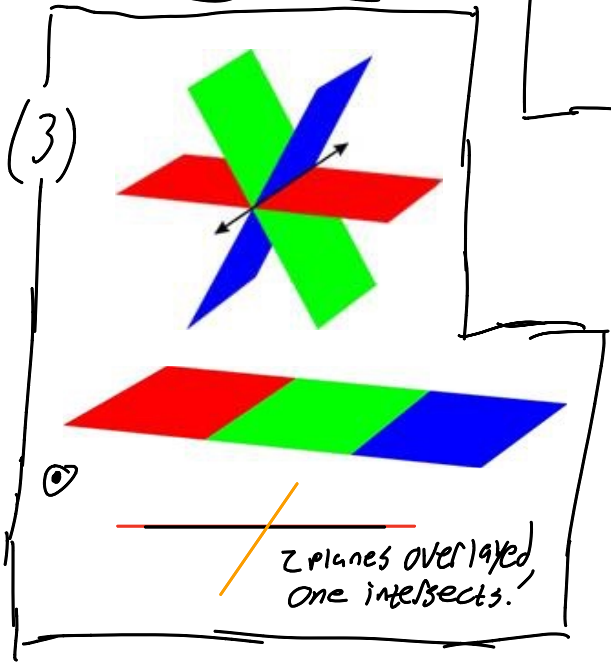
(1)



2)



(3)



(A) No soln case: $\begin{cases} 5x-2y+z=3 \\ 4x-4y-8z=2 \\ -x+y+2z=-3 \end{cases}$ *

(B) ∞ soln case: $\begin{cases} 2x+2y+2z=-2 \\ 2x+3y+2z=4 \\ x+y+z=-1 \end{cases}$ ⊙

$$\left\{ \begin{array}{l} \det(A) = \begin{vmatrix} 5 & -2 & 1 \\ 4 & -4 & -8 \\ -1 & 1 & 2 \end{vmatrix} = 0 \\ \\ \det(B) = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0 \end{array} \right.$$

What's an easy way to tell?
Remember: these 3 eqns, geometrically describe 3 planes! Coeffecients are normal vectors.

If normal vectors are some linear combination of one another, A^{-1} DNE and there is either ∞ solns or no soln at all!

A^{-1} DNE means $\det(A) = 0$

(A) No soln case: $\begin{cases} 5x-2y+z=3 \\ 4x-4y-8z=2 \\ -x+y+2z=-3 \end{cases}$: $\left[\begin{array}{ccc|c} 5 & -2 & 1 & 3 \\ 4 & -4 & -8 & 2 \\ -1 & 1 & 2 & -3 \end{array}\right]$

$r_2 = -4r_3$
(Coeffecients)

$r_2 : r_2 + 4r_3$ $\left[\begin{array}{ccc|c} 5 & -2 & 1 & 3 \\ 0 & 0 & 0 & -10 \\ \cdots & & & \cdots \end{array}\right]$ $0 = -10$?? FALSE. $\emptyset$.

(B) ∞ soln case: $\begin{cases} 2x+2y+2z=-2 \\ 2x+3y+2z=4 \\ x+y+z=-1 \end{cases} \sim \left[\begin{array}{ccc|c} 2 & 2 & 2 & -2 \\ 2 & 3 & 2 & 4 \\ 1 & 1 & 1 & -1 \end{array}\right] \overset{r_1 : r_1 - 2r_3}{\sim} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 3 & 2 & 4 \\ 1 & 1 & 1 & -1 \end{array}\right]$

$\sim r_2 : r_2 - 2r_3 \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 1 & 1 & 1 & -1 \end{array}\right] \sim r_3 : r_3 - r_2 \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 1 & -7 \end{array}\right]$

$0 = 0$ true

$$\boxed{\begin{array}{c} y = 6 \\ x + z = -7 \end{array}}$$

(Ch3) Inverse Matrices

$\Rightarrow$ The idea: $2(1)=2,\ 3(1)=3, \ldots$

$$AI = A, \quad BI = B$$
$$IA = A, \quad IB = B$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3\times 3 \text{ case}).$$

$$\Rightarrow 3\cdot\frac{1}{3} = 1 \rightarrow \boxed{AA^{-1} = I = A^{-1}A}$$

$$\Rightarrow \text{System: } A\vec{x} = \vec{b} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow I\vec{x} = A^{-1}\vec{b}$$
$$\underline{\vec{x} = A^{-1}\vec{b}}$$

Ex

$$2x + y + z = 0$$
$$x + 2y + z = 2$$
$$-x + y - z = 1$$

$$\underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & -1 \end{bmatrix}} \underset{\vec{x}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} \underset{=}{=} \underset{\vec{b}}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}$$

$$\Rightarrow \vec{x} = A^{-1}\vec{b}$$

Find A^{-1}

structure: $$\left[\begin{array}{c:c} A & I \end{array}\right] \xrightarrow{\text{row reduct.}} \left[\begin{array}{c:c} I & A^{-1} \end{array}\right]$$

Why? Don't Do this in Class, but good to have!

$$AA^{-1} = \begin{bmatrix}1&3&1\\2&3&-1\\-4&0&2\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$\begin{bmatrix}1&3&1\\2&3&-1\\-4&0&2\end{bmatrix}\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ → putting left side as $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ forces right side to be $\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix}$

$\begin{bmatrix}1&3&1\\2&3&-1\\-4&0&2\end{bmatrix}\begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ → putting left side as $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ forces right side to be $\begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix}$

$\begin{bmatrix}1&3&1\\2&3&-1\\-4&0&2\end{bmatrix}\begin{bmatrix}a_{13}\\a_{23}\\a_{33}\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ → putting left side as $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ forces right side as $\begin{bmatrix}a_{13}\\a_{23}\\a_{33}\end{bmatrix}$

So, we may do this all at once — in parallel:

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\2&3&-1&0&1&0\\-4&0&2&0&0&1\end{array}\right]$$

↑ go to I ; ↑ will be A^{-1}

$$A = \begin{bmatrix}2&1&1\\1&2&1\\-1&1&-1\end{bmatrix}, \quad A^{-1}: \left[\begin{array}{ccc|ccc}2&1&1&1&0&0\\1&2&1&0&1&0\\-1&1&-1&0&0&1\end{array}\right]$$

$$\underset{R_2:\,R_2+R_3}{\overset{R_1:\,R_1+R_3}{\sim}} \left[\begin{array}{ccc|ccc}1&2&0&1&0&1\\0&3&0&0&1&1\\-1&1&-1&0&0&1\end{array}\right] \underset{R_3:\,R_3+R_1}{\overset{R_2/R_3}{\sim}} \left[\begin{array}{ccc|ccc}1&2&0&1&0&1\\0&1&0&0&\frac{1}{3}&\frac{1}{3}\\0&3&-1&1&0&2\end{array}\right]$$

$$\sim \begin{matrix} R_1: R_1 - 2R_2 \\ R_3: R_3 - 3R_2 \end{matrix} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right]$$

$$\overset{-R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/3 & 1/3 \\ 0 & 1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] \Longrightarrow A^{-1} = \begin{bmatrix} 1 & -2/3 & 1/3 \\ 0 & 1/3 & 1/3 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\vec{X} = A^{-1}\vec{b}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2/3 & 1/3 \\ 0 & 1/3 & 1/3 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

An important formula: For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\Longrightarrow \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Verify: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $\quad A^{-1} = \dfrac{1}{-4}\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$

$$I = A^{-1}A = \frac{-1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$= I$$

Where does it come from?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} ax+bz &= 1 \\ ay+bw &= 0 \\ cx+dz &= 0 \\ cy+dw &= 1 \end{aligned}\right\}$$

4 eqns

4 unknowns (x,y,z,w)

Solveable!!

(Ch.14) Linear Transformations

Here's an idea: $c\vec{v}$ scales a vector; what if we wanted to move a vector, or rotate the vector? Transform the vector to a new vector.

$\Longrightarrow$ Matrices tell vectors how to transform

$$A\vec{v} = \vec{u}$$

How?

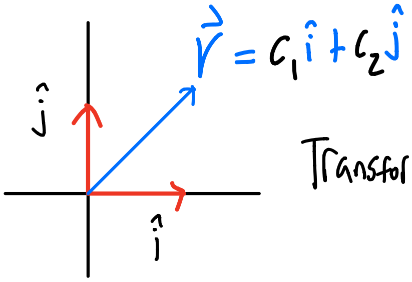
Any vector in 2D: $\vec{v} = x\hat{i} + y\hat{j}$

All vectors in 2D are "made" by simply changing x or y.

$\vec{v}$ spans $\mathbb{R}^2$ (given $x, y \neq 0$).

Idea: If we transform $\hat{i}, \hat{j}$, the basis for $\mathbb{R}^2$, we effectively transform all vectors in $\mathbb{R}^2$ (b/c all vectors have $\hat{i}, \hat{j}$ as fundamental ingredients)

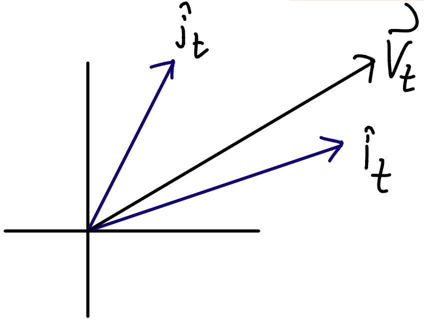
Let $\vec{v} = c_1\hat{i} + c_2\hat{j}$



$$A\vec{v} = c_1(A\hat{i}) + c_2(A\hat{j})$$

Transformed $\vec{v} = c_1(\text{transformed } \hat{i}) + c_2(\text{transformed } \hat{j})$

$$\vec{v}_t = c_1\hat{i}_t + c_2\hat{j}_t$$



Ex | Find the matrix A that stretches the horizontal axis 3x and compresses the vertical axis by a factor of 2.

$A\vec{V}$ will stretch the horizontal comp. of $\vec{V}$ 3x and compress the vertical comp. of $\vec{V}$ by a factor of 2.

$$A\vec{V} = c_1\hat{i}_t + c_2\hat{j}_t = c_1\begin{bmatrix}3\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1/2\end{bmatrix} = \begin{bmatrix}3 & 0\\0 & 1/2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$A = \begin{bmatrix}3 & 0\\0 & 1/2\end{bmatrix} = \left[\begin{array}{c|c}\hat{i}_t & \hat{j}_t\end{array}\right]$$

Columns of A tell you where to send $\hat{i}, \hat{j}, \hat{k}, \ldots$.

Ex Find the matrix A that

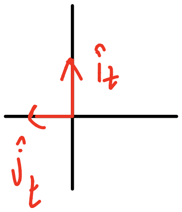
A ① Shears the vertical axis to the line $y = 2x$
B ② Rotates the plane CCW by 90°.

$\Longrightarrow$ Break apart into 2 matrices, then multiply together.

Why? $\quad A\vec{v} = v_t$

$$B\vec{v}_t = \vec{v}_{final} = BA\vec{v} = (②)(①)\,\vec{v}\ ;\ \boxed{① \to ② = (②)(①)}$$

$$B:\quad B = \begin{array}{cc} \hat{i}_t & \hat{j}_t \\ \left[\begin{matrix} 0 \\ 1 \end{matrix}\right. & \left.\begin{matrix} -1 \\ 0 \end{matrix}\right] \end{array}$$



A: In $\mathbb{R}^2$, an important property of shear for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is that $ad - bc = 1$. This will make more sense later.

$$y = 2x,\quad x = \tfrac{1}{2},\quad y = 1,\quad A = \begin{array}{cc} \hat{i}_t & \hat{j}_t \\ \left[\begin{matrix} 1 \\ 0 \end{matrix}\right. & \left.\begin{matrix} 1/2 \\ 1 \end{matrix}\right] \end{array}$$

$$① \to ② = (②)(①) = BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 \\ 1 & 1/2 \end{bmatrix}}$$

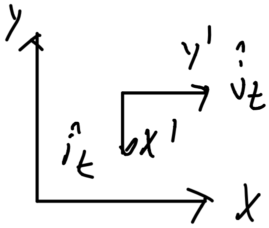
Don't Do in Recitation

Extra Exercise: Find the matrix that undoes the above tranformation.

① Rotate CW 90°

② Send the line $y=2x$ to the vertical (Send the y-axis to $y=-2x$) via a shear.

①: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



②: $\begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}$

① → ② = (②)(①) = $\begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\boxed{A' = \begin{bmatrix} 1/2 & 1 \\ -1 & 0 \end{bmatrix}}$$

Remember, the last problem: $A = \begin{bmatrix} 0 & -1 \\ 1 & 1/2 \end{bmatrix}$

$$AA' = \begin{bmatrix} 0 & -1 \\ 1 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A' = A^{-1}$ (expected)

(Ch 15) Coordinate Transformations

Idea: $\hat{i}$ and $\hat{j}$ span all of 2-D space. For any $\vec{V} = \vec{V}(x,y)$;

$$\vec{v} = x\hat{i} + y\hat{j}$$

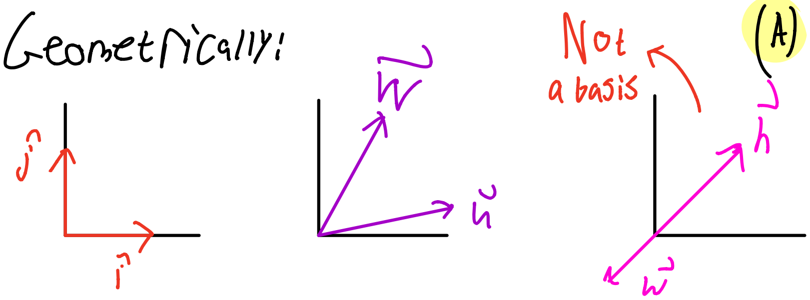
$\hat{i}$ and $\hat{j}$ are the fundamental building blocks of $\mathbb{R}^2$. And remember; $\hat{i}$ and $\hat{j}$ are vectors themselves!

=> Are there other vectors which can define all of 2D space?

Is there another BASIS to $\mathbb{R}^2$?

Yes! There are an infinite # of Bases.

Geometrically:

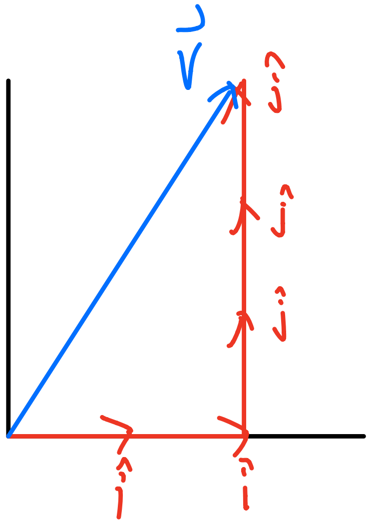


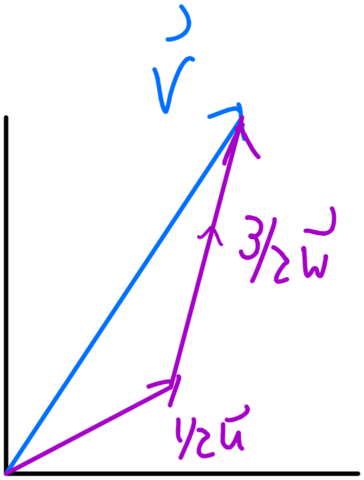
(A) $c_1\vec{h} + c_2\vec{w} = \vec{0}$

True for $c_1 = c_2 \neq 0$

$$\vec{V} = 2\hat{i} + 3\hat{j}$$
$$\vec{V} = \tfrac{1}{2}\vec{u} + \tfrac{3}{2}\vec{w}$$
} Example

$\vec{h}, \vec{w}$ are LI iff the only #'s satisfying (A) are $c_1 = c_2 = 0$. But that's not the case.





Ex For a point (x,y) in the basis $\{\hat{i},\hat{j}\}$, find the same Point (a,b) in the basis $\{\vec{V}_1,\vec{V}_2\}$:

$$\vec{V}_1=\begin{bmatrix}1\\1\end{bmatrix},\quad \vec{V}_2=\begin{bmatrix}2\\1\end{bmatrix}$$

variables

$\Rightarrow$ This means: For a vector $\vec{V}=\overset{\downarrow}{x}\,\hat{i}+\overset{\downarrow}{y}\,\hat{j}$, Find the corresponding a,b such that $\vec{V}=a(x,y)\vec{V}_1+b(x,y)\vec{V}_2$.

$$\vec{V}=a\vec{V}_1+b\vec{V}_2=a\begin{bmatrix}1\\1\end{bmatrix}+b\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

$$\vec{V}=x\hat{i}+y\hat{j}=\begin{bmatrix}x\\y\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

$$A\quad \vec{X}\ =\ \vec{b}$$

$$\vec{X}=A^{-1}\vec{b}$$

$$A^{-1}=\frac{1}{1-2}\begin{bmatrix}1&-2\\-1&1\end{bmatrix}=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}$$

$$\vec{X}=A^{-1}\vec{b}$$

$$\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-x+2y\\x-y\end{bmatrix}\Big\}\ \boxed{\begin{array}{l}a=2y-x\\ \hline b=x-y\end{array}}$$

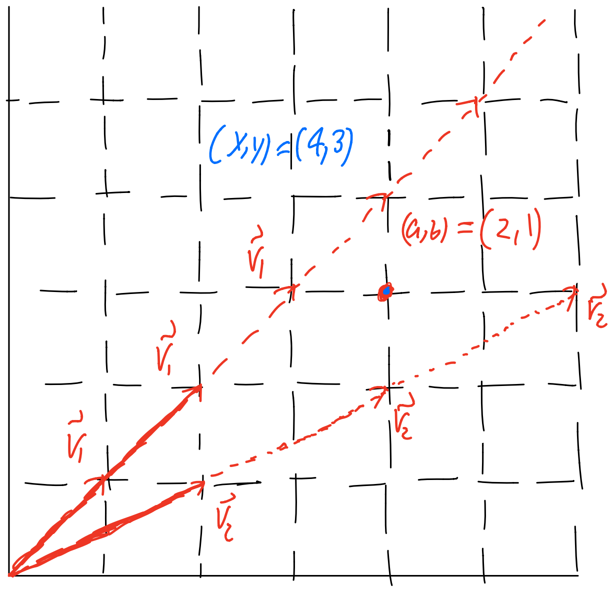
$$\Longrightarrow \boxed{\vec{V}=x\hat{i}+y\hat{j}=a\vec{V}_1+b\vec{V}_2=(2y-x)\vec{V}_1+(x-y)\vec{V}_2}$$

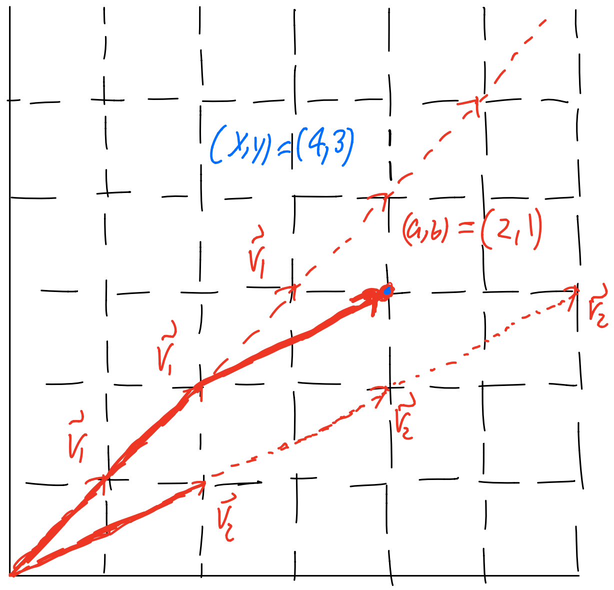
So: If $x=4$, $y=3$; $\vec{v} = 4\hat{i} + 3\hat{j} = 4\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$

$$\vec{v} = (2\cdot 3 - 4)\vec{v_1} + (4-3)\vec{v_2} = 2\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$$ ✓

$$= 2\vec{V_1} + 1\vec{V_2} = \begin{bmatrix}4\\3\end{bmatrix}$$

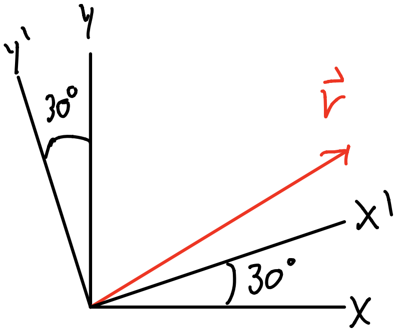
What does this look like?





Ex | Another Coordinate Transformation problem

Write the vector $\vec{v} = \begin{bmatrix}4\\3\end{bmatrix}$ in the basis with the x-y plane rotated 30° CCW.



$$\vec{v} = 4\hat{i}+3\hat{j} = c_1\underbrace{\hat{i}_t + c_2\hat{j}_t}_{\text{in } x'\text{-}y' \text{ frame}}$$

Goal: Solve for c_1, c_2, these will be the coordinates of $\vec{v}$ in the prime frame.

First: Determine $\hat{i}_t, \hat{j}_t$.

By the rotation matrix: $R_\theta = \left[\begin{array}{cc} \hat{i}_t & \hat{j}_t \\ \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$

$$R_{30^\circ} = \begin{bmatrix}\sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2\end{bmatrix}$$

$$\vec{v} = 4\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = c_1\begin{bmatrix}\sqrt{3}/2\\1/2\end{bmatrix} + c_2\begin{bmatrix}-1/2\\\sqrt{3}/2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}\sqrt{3}/2 & -1/2\\1/2 & \sqrt{3}/2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}\sqrt{3}/2 & -1/2\\1/2 & \sqrt{3}/2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$\vec{V} = A\vec{X}$$

$$\vec{X} = A^{-1}\vec{V} = \begin{bmatrix}\sqrt{3}/2 & 1/2\\-1/2 & \sqrt{3}/2\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2\sqrt{3}+3/2\\-2+\frac{3\sqrt{3}}{2}\end{bmatrix}$$

$$\boxed{\vec{v} = \begin{bmatrix}2\sqrt{3}+3/2\\\frac{3\sqrt{3}}{2}-2\end{bmatrix} \text{ in the } x'\text{-}y' \text{ basis.}}$$

(Ch16/17) Determinants: Geometrically Speaking, the determinant gives the n-dimensional oriented Volume of the parallelopiped spanned by n-dimensional vectors.

Algebraically, the determinant is a little strange.

Ex Compute $\det\begin{bmatrix}1&0&3\\3&1&4\\0&2&1\end{bmatrix}$. Expand along row 1 and column 1.

Row 1 expansion: $1\begin{vmatrix}1&4\\2&1\end{vmatrix} - 0\begin{vmatrix}3&4\\0&1\end{vmatrix} + 3\begin{vmatrix}3&1\\0&2\end{vmatrix} = -8+3(6) = 10$

Col 1 expansion: $1\begin{vmatrix}1&4\\2&1\end{vmatrix} - 3\begin{vmatrix}0&3\\2&1\end{vmatrix} + 0\begin{vmatrix}0&3\\1&4\end{vmatrix} = -8-3(-6) = 10$ ✓

General pattern for expansion: $\begin{bmatrix}+&-&+&-&\cdots&-\\-&+&-&+&\cdots&-\\+&-&+&-&\cdots&\vdots\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\end{bmatrix}$

Remember this!

Ex What is the area spanned by the vectors $\vec u = \begin{bmatrix}3\\1\\3\end{bmatrix}$, $\vec v = \begin{bmatrix}4\\1\\4\end{bmatrix}$

$$\text{Area} = \|\vec u\times\vec v\|;\quad \det\begin{bmatrix}\hat i&3&4\\\hat j&1&1\\\hat k&3&4\end{bmatrix} = \begin{vmatrix}1&1\\3&4\end{vmatrix}\hat i - \begin{vmatrix}3&4\\3&4\end{vmatrix}\hat j + \begin{vmatrix}3&4\\1&1\end{vmatrix}\hat k$$

$$= 1\hat i - 0\hat j - 1\hat k$$

$$= \hat i - \hat k$$

$$\Rightarrow \|\hat i - \hat k\| = \sqrt{1^2+(-1)^2} = \boxed{\sqrt{2}}$$

Skip: Why this works; $\vec u\cdot(\vec v\times\vec w) = \det[\vec u|\vec v|\vec w]$. $(\hat i,\hat j,\hat k)\cdot(\vec v\times\vec w)$

$= (\vec v\times\vec w)_{\hat i}\,\hat i + (\vec v\times\vec w)_{\hat j}\,\hat j + (\vec v\times\vec w)_{\hat k}\,\hat k$

Ex $\det\left(\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}\right)$

$$\det(ABC) = \det(A)\det(B)\det(C)$$

$$= (3)(3)(5) = \boxed{45}$$

Ex $$\det\begin{array}{c} \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 1 & 4 & 0 \\ 3 & 1 & 6 & 1 \\ 4 & 4 & 8 & 2 \end{bmatrix} \\ \begin{matrix} C_1 & & C_3 & \end{matrix} \end{array} = 0.$$

Why? $C_3 = 2C_1$

These vectors $[C_1]$, $[C_3]$

Are colinear, therefore

the 4-d volume spanned

by these vectors is zero!

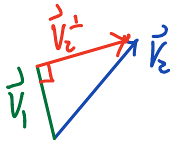
We can prove that the determinant of an nxn matrix gives the n-dim volume of an n-dim parallelipiped using gram-schmidt orthogonalization.

We start w/ the set of vectors $\{r_1, \dots, r_n\}$ so that

Here's the idea:

$$\vec{V}_1 = \vec{r}_1$$

$$\vec{V}_2 = \underbrace{c}_{\text{multiple of } \vec{V}_1}\vec{V}_1 + \underbrace{\vec{V}_2^{\perp}}_{\text{vector} \perp \text{to } \vec{V}_1}$$



Do this for each vector:

$$\vec{V}_1 = \vec{r}_1$$

$$\vec{V}_2 = C_{12}\vec{V}_1 + \vec{V}_2^{\perp}$$

$\vec{V}_2^{\perp}$ is ⊥ to $\vec{V}_1$

$$\vec{V}_3 = C_{13}\vec{V}_1 + C_{23}\vec{V}_2 + \vec{V}_3^{\perp}$$

$\vec{V}_3^{\perp}$ is ⊥ to plane of $\vec{V}_1, \vec{V}_2$

$$\vec{V}_4 = C_{14}\vec{V}_1 + C_{24}\vec{V}_2 + C_{34}\vec{V}_3 + \vec{V}_4^{\perp}$$

$\vec{V}_4^{\perp}$ is ⊥ to hyperplane of $\vec{V}_1, \vec{V}_2, \vec{V}_3$

$$\vdots$$

Notice: $\det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = \det(\vec{r}_1, c_{12}\vec{v}_1 + \vec{v}_2^{\perp}, c_{13}\vec{v}_1 + c_{23}\vec{v}_2 + \vec{v}_3^{\perp}, \dots)$

How can we simplify this?

Suppose vectors in $\mathbb{R}^2$:

$$\det(\vec{a}, \vec{b}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$$

$$\downarrow \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \downarrow$$

$$\det(\vec{a} + \vec{c}, \vec{b}) = \begin{vmatrix} a_1 + c_1 & b_1 \\ a_2 + c_2 & b_2 \end{vmatrix} = (a_1 + c_1)b_2 - (a_2 + c_2)b_1 = (a_1 b_2 - a_2 b_1) + (c_1 b_2 - c_2 b_1)$$

$$= \det(\vec{a}, \vec{b}) + \det(\vec{c}, \vec{b})$$

And $\det(\vec{a} + \vec{b} + \vec{c}, \vec{d} + \vec{e}) = \det(\vec{a}, \dots)$

[yes proof incomplete, but this should get you pretty far! See if you can do it yourself from here!].

(Ch 18) Computing determinants With Row Reduction

Idea: $\det\Big(\overbrace{\text{triangular matrix}}^{A}\Big) = \text{Prod. of diagonals}$ $\quad \det\begin{bmatrix}1&2&3\\0&4&7\\0&0&2\end{bmatrix} = 8$

$$\det(A^T) = \det\begin{bmatrix}1&0&0\\2&4&0\\3&7&2\end{bmatrix} = 8$$

We can reduce any nxn matrix into triangular form in order to compute its determinant. However, we must be careful!

Row operations: From $A \xrightarrow{\text{operation}} A'$ $\qquad \vec{u}\cdot(\vec{v}\times\vec{w}) = -\vec{u}\cdot(\vec{w}\times\vec{v})$

① Interchange 2 rows: $\det(A') = -\det(A)$ (reverses orientation)

② mult row by const c: $\det(A') = c\det(A)$ (scaling sidelength by c)

③ Add mult. one row to another: $\det(A') = \det(A)$ (Shear)

Result: $\operatorname{Det}(A) = \operatorname{Det}(A^T)$ in general!

How? Exchange matrix: $\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix} = A_1$; $\det(A_1) = -1$, $A_1^T = \begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix} \Rightarrow \det(A_1^T) = -1$

$R_1 \longleftrightarrow R_4$ $\qquad \det(A_1) = \det(A_1^T)$

Rescale matrix: $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&c&0\\0&0&0&1\end{bmatrix} = A_2$; $\det(A_2) = c$, $A_2^T = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&c&0\\0&0&0&1\end{bmatrix} \Rightarrow \det(A_2^T) = c$

$\det(A_2) = \det(A_2^T)$

$R_3 \to cR_3$

Combine matrix: $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&c&0&1\end{bmatrix} = A_3$, $\det(A_3) = \det(A_3^T) = 1$

$R_4: R_4 + cR_2$ $\qquad \Rightarrow \det(A_{1,2,3}) = \det(A_{1,2,3}^T)$

$\longrightarrow$ So: If U is the triangular matrix obtained after performing row operations to A; Ex: $U = A_3 A_1 A_2 A$

$= \det(U) = \det(A_3)\det(A_1)\det(A_2)\det(A)$

We know $U^T = (A_3A_1A_2A)^T = A^TA_2^TA_1^TA_3^T$

switch rows + columns.

Proof:

Recall $(AB)_{ij} = \sum_{k=1}^{\infty} A_{ik}B_{kj}$

$\uparrow$ i^{th} row j^{th} col entry of AB

$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki}$

$\|$

$(B^TA^T)_{ij} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = \sum_{k=1}^{n} B_{ki}A_{jk}$

$\Rightarrow (AB)^T = B^TA^T$

$\Longrightarrow \det(U^T) = \det(A^TA_2^TA_1^TA_3^T) = \det(A^T)\det(A_2^T)\det(A_1^T)\det(A_3^T)$

$\Longrightarrow \det(U) = \det(U^T)$ (triangular)

$$\Longrightarrow \det(A^T) = \frac{\det(U^T)}{\det(A_2^T)\det(A_1^T)\det(A_3^T)} = \frac{\det(U)}{\det(A_2)\det(A_1)\det(A_3)}$$

$= \det(A)$

$$\Longrightarrow \boxed{\det(A) = \det(A^T)}$$

Ex $\det\begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix} = a^2(3-a)$

Show this is true w/ row operations.

Goal: Make the matrix triangular. So lets start by cleaning up column 1.

$$\begin{matrix} r_2 : r_2 - r_3 \\ r_1 : r_1 - r_3 \end{matrix} \begin{bmatrix} -a & 0 & a \\ 0 & -a & a \\ 1 & 1 & 1-a \end{bmatrix}$$

↗ There appears to be more opportunity to clear in the columns, so take the transpose! $\longrightarrow \det(A) = \det(A^T)$

$$\Longrightarrow \begin{bmatrix} -a & 0 & 1 \\ 0 & -a & 1 \\ a & a & 1-a \end{bmatrix}$$

Take it col. by col.

$$r_3 : r_3 + r_1 \begin{bmatrix} -a & 0 & 1 \\ 0 & -a & 1 \\ 0 & a & 2-a \end{bmatrix} \underset{\sim}{R_3 : R_3 + R_2} \begin{bmatrix} -a & 0 & 1 \\ 0 & -a & 1 \\ 0 & 0 & 3-a \end{bmatrix}$$

$$\Longrightarrow \det(A^T) = \det(A) = (-a)(-a)(3-a)$$

$$= \underline{a^2(3-a)} \checkmark$$

Vol. 2 Derivatives

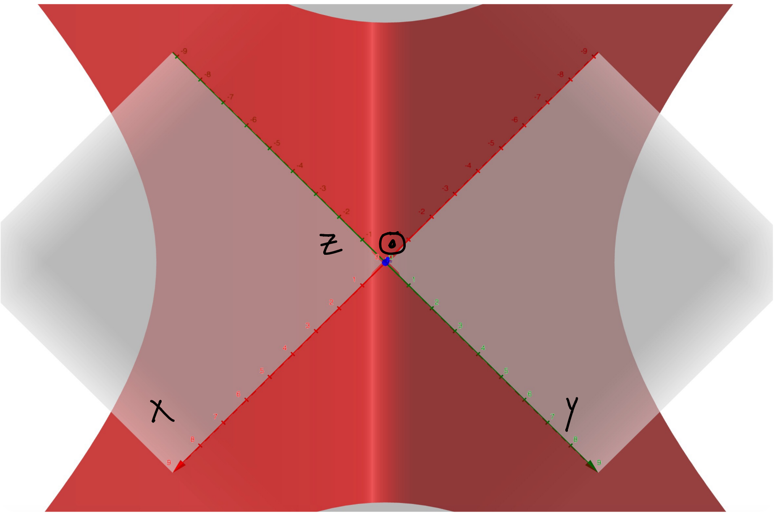
(Ch 1) Multivariate Functions

Let's consider a simple case: $f(x,y) = xy$

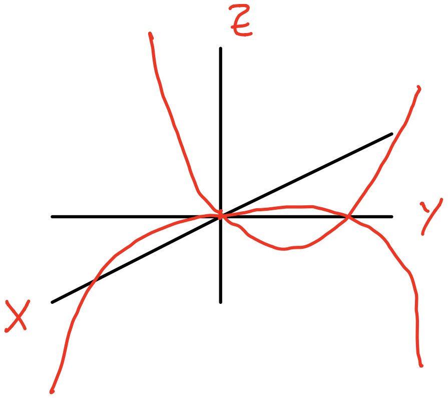
f: Takes inputs in $\underline{\mathbb{R}^2}$ and maps to outputes in $\underline{\mathbb{R}}$.

$\Longrightarrow$ 2 inputs for every one output.

How to graph: $z = f(x,y)$



Along the line $y=x$ it behaves like a quadratic:



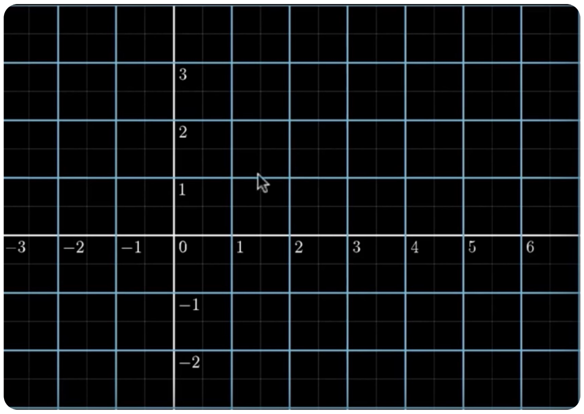
Along line $y=-x$ in the plane it behaves like a downwards opening quadratic.

$\longrightarrow$ Called a SADDLE

What does a function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ look like?

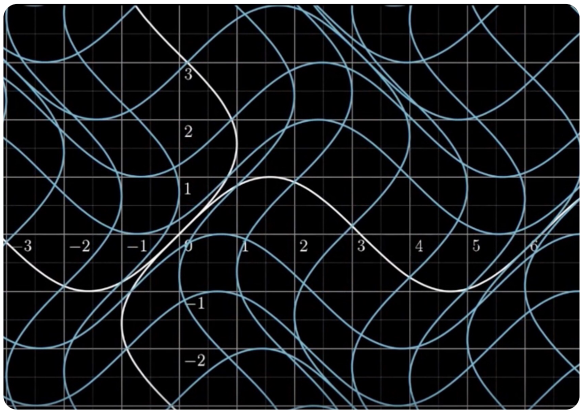
$$f\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+\sin y\\y+\sin x\end{bmatrix}$$

- This says: take a point (x,y) in the xy plane and move it to a new set of coordinates $(x+\sin y, y+\sin x)$



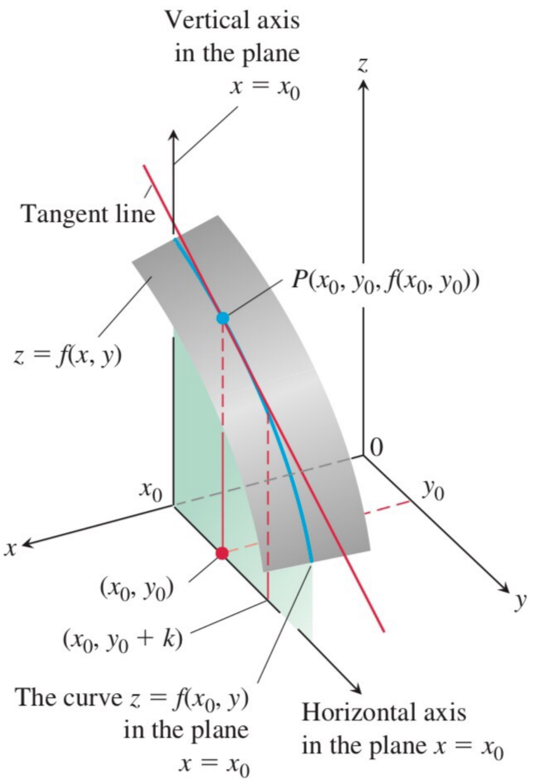
Blue grid has pts. (x,y)

$$\downarrow \; f\begin{bmatrix}x\\y\end{bmatrix} \; \downarrow$$



$(x_0, y_0) \longrightarrow (x_0+\sin y_0, y_0+\sin x_0)$ via f.

(Ch2) Partial Derivatives



Given $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$ Hold y Const.

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$ Hold X Const.

Ex | Compute all partials of

$f(x,y,z) = (x^2y^3)^{2z}$ at $x=y=z=1$

$$\Longrightarrow \frac{\partial f}{\partial x}\Big|_1 = 2z(x^2y^3)^{2z-1}(2xy^3)\Big|_1 = 4$$

$$\Longrightarrow \frac{\partial f}{\partial y}\Big|_1 = 2z(x^2y^3)^{2z-1}(x^2(3y^2))\Big|_1 = 6$$

$$\Longrightarrow \frac{d}{dx} a^x = a^x \ln a$$

$$\frac{\partial f}{\partial z}\Big|_1 = (x^2y^3)^{2x} \ln(x^2y^3) \cdot 2\Big|_1 = 0$$

(Ch3) The derivative matrix

Here's the idea: We've seen that a multivariate function can have many partial derivatives (one for each variable), so we can use a matrix of partials to capture all of the derivative information.

Call this matrix $[Df]$.

What is the size of $[Df]$? Well, it depends on the size of the function!

Suppose you have a function $f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix}$

$$[Df] = \begin{array}{c|cccc} & \partial/\partial x_1 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \cdots \\ \hline u_1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots \\ u_2 & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots \\ u_3 & \frac{\partial u_3}{\partial x_3} & \frac{\partial u_3}{\partial x_3} & \frac{\partial u_3}{\partial x_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

Output information in rows

Input information in columns

Pattern emerges: Size of $[Df]$ = # outputs × # inputs

Ex $f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 8x^2y + z^2 \\ 18 + 3y + z^{3/2} \end{bmatrix}$ Compute $[Df]$.

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$. 3 inputs, 2 outputs

$\Longrightarrow$ size$[Df] = 2 \times 3$

$$\boxed{[Df] = \begin{array}{c|ccc} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \hline u & 16xy & 8x^2 & 2z \\ p & 0 & 3 & \frac{3}{2}z^{1/2} \end{array}}$$

At the point $(x,y,z) = (1,2,-1)$, which inputs are the outputs most sensitive to?

$$[Df]_{(-1,2,1)} = \begin{array}{ccc|c} x & y & z & \\ \hline -32 & 8 & 2 & u \\ & & & \\ 0 & 3 & \frac{3}{2} & p \end{array}$$

$\boxed{\begin{array}{l} u \text{ is most sensitive to } \Delta X \\ P \text{ is most sensitive to } \Delta Y \end{array}}$

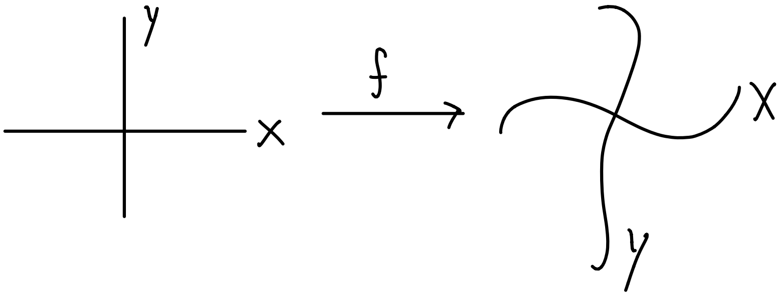
But What does it truly mean? What is [Df]??

This is Something you might not get in lecture but Will be a great Conceptual aide in the future:

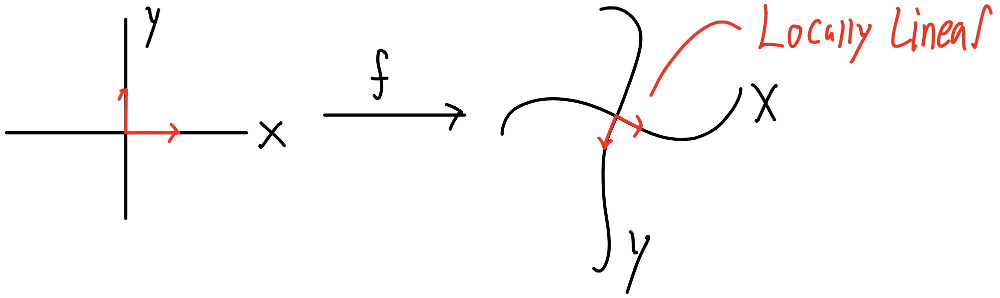
$$\Rightarrow f\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}f_1\\f_2\end{bmatrix} \to [Df] = \begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix} \longleftarrow \text{Linear Transformation}$$

Remember; matrices are linear transformations. So what sort of linear transformation is this?

The answer: $f\begin{bmatrix}x\\y\end{bmatrix}$ Will usually warp space in a very non-linear way! It Sends



This is clearly non-linear. But if you zoom into a region, it is locally linear (remember local linearity?)



f Behaves as if it is a linear transformation, if we only consider very small regions. So what is that locally linear transformation? [Df] !!!

$$[Df] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

Take small step in y dir, record how f_1, f_2 change

Take a small step in x dir, record how f_1, f_2 change

Remember: $f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

[Df] Tells use what f does to a small region of space! It is the locally linear transformation induced by the non-linear function f.

If you remember this now, things which seem complex and unobvious will become crystal clear later.

Well then: If we have the ROC of inputs vector $\vec{h}$, what do we get when we transform $\vec{h}$ (geometrically speaking):

$$[Df]_{\vec{a}} \vec{h}_{\vec{a}}$$

$\Longrightarrow [Df]_{\vec{q}}\,\vec{h}_{\vec{q}}$ maps motion in input space (x,y plane) to the output space (f_1, f_2 plane). Thus, we now have the rate of change of outputs expressed as a vector.

Running one way in the input space causes you to run another way in the output space.
That new direction is the RoC outputs $= [Df]_{\vec{q}}\,\vec{h}_{\vec{q}}$
Plus, f just transforms vectors (locally), so
RoC inputs → RoC of outputs via [Df], since [Df] tells use how f transforms small regions of space.

Algebraically: $\vec{h} = \begin{bmatrix} \partial x/\partial t \\ \partial y/\partial t \end{bmatrix}$

Implicit Differention: x(t), y(t).

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \begin{bmatrix} \partial x/\partial t \\ \partial y/\partial t \end{bmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial t} \\ \frac{\partial f_2}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_2}{\partial y}\frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \partial f_1/\partial t \\ \partial f_2/\partial t \end{pmatrix} = \vec{L}$$

$$\boxed{\vec{L} = [Df]_{\vec{q}}\,\vec{h}}$$

Ex If $f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\sin(xy) \\ 3yz^2 + x \end{bmatrix}$ and the ROC of inputs at $(1,1,1)$ is $(\dot{x}, \dot{y}, \dot{z}) = (1,2,0)$; What is the ROC of outputs?

$$\vec{L} = [Df]_{\vec{a}}\,\vec{h}.$$

$$\vec{L} = ??$$

$$\vec{h} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$[Df] = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array}$$

$$[Df] = \begin{bmatrix} 2y\cos(xy) & 2x\cos(xy) & 0 \\ 1 & 3z^2 & 6yz \end{bmatrix}$$

$$[Df]_{(1,1,1)}\,\vec{h} = \begin{bmatrix} 2\cos(1) & 2\cos(1) & 0 \\ 1 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{L} = \begin{bmatrix} 6\cos(1) \\ 7 \end{bmatrix}}$$

Ex If the rate of change of outputs for a function are $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and

$[Df] = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ What is the ROC of inputs?

$$\vec{L} = [Df]\vec{h}$$

$$\vec{h} = [Df]^{-1}\vec{L} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 12 \\ -7 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ -7/3 \end{bmatrix}}$$

Ex Great problem: Given that the derivative of $f:\mathbb{R}^2\to\mathbb{R}^2$ at a particular input satisfies

$[Df]\begin{bmatrix}2\\-3\end{bmatrix}=\begin{bmatrix}-3\\5\end{bmatrix}$, then how fast are the outputs changing at that input if the inputs are changing w/ rates 6 and -9 respectively?

$$[Df]=\left[\frac{\partial\vec{f}}{\partial x}\,\middle|\,\frac{\partial\vec{f}}{\partial y}\right]=\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix}=2\begin{bmatrix}a\\c\end{bmatrix}-3\begin{bmatrix}b\\d\end{bmatrix}=\begin{bmatrix}-3\\5\end{bmatrix}$$

mult. both sides by 3: $6\begin{bmatrix}a\\c\end{bmatrix}-9\begin{bmatrix}b\\d\end{bmatrix}=\begin{bmatrix}-9\\15\end{bmatrix}$

$$=\underbrace{\begin{bmatrix}a&b\\c&d\end{bmatrix}}_{[Df]}\underbrace{\begin{bmatrix}6\\-9\end{bmatrix}}_{\vec{h}}=\underbrace{\boxed{\begin{bmatrix}-9\\15\end{bmatrix}}}_{\vec{L}}$$

(Ch4) $[Df]$ behaves as a sort of derivative for functions of arbitrary dimension. So how do we define differentiability for the derivative?

Recall single variable: $f(x+\Delta x) \approx f(x) + \frac{\partial f}{\partial x}\Delta x + \ldots + O(x^2)$

Becomes exact in lim as $\Delta x \to 0$

$$\lim_{\Delta x \to 0}\left[f(x+\Delta x) - f(x) = \frac{\partial f}{\partial x}\Delta x\right]$$

Divide both sides by Δx:

$$\lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\partial f}{\partial x}\right]$$

$$\lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{\partial f}{\partial x}\cdot\frac{\Delta x}{\Delta x} = 0\right]$$

$$\lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x) - \frac{\partial f}{\partial x}\Delta x}{\Delta x} = 0\right]$$

This must be true if f is to be differentiable.

The same can be done with $[Df]$, starting with

$f(\vec{a}+\vec{h}) = f(\vec{a}) + [Df]_{\vec{a}}\vec{h} + o(\|\vec{h}^2\|)$ as $\|\vec{h}\| \to 0$

$$\longrightarrow \lim_{\|\vec{h}\| \to 0} \frac{\overbrace{(f(\vec{a}+\vec{h}) - f(\vec{a}))}^{\partial f} - \overbrace{[Df]_{\vec{a}}\vec{h}}^{\partial f}}{\|h\|} = \vec{0}$$

Hard example: $f(A) = A^{-1}$. Use the def. of deriv. to compute $[Df]$ at $A = I$ (identity). Assume A is square.

$$f(A+H) = f(A) + [Df]_A H + o(|H|^2) \quad \text{in } \lim |H| \to 0$$

$$f(A+H) - f(A) = [Df]_A H + o(|H|^2)$$

$$f(I+H) - f(I) = (I+H)^{-1} - I = (I+H)^{-1} - (I+H)^{-1}(I+H)$$

$$= (I+H)^{-1}(I - (I+H))$$

$$= (I+H)^{-1}(I - I - H)$$

$$= (I+H)^{-1}(-H)$$

$$= -(I+H)^{-1}H$$

We may suppose : $I + H \approx I$ in $\lim |H| \to 0$

$$\Rightarrow f(I+H) - f(I) = -IH$$

$$\boxed{[Df]_I = -I}$$

(Ch5) The Chain Rule

We Know: $\frac{d}{dx}(f\circ g) = \frac{d}{dx}(f(g(x)) = f'(g(x))\,g'(x)$

Evaluate at $x=a$: $f'(g(a))g'(a)$

The same works for $[D(f\circ g)]_{\vec a} = [Df]_{g(a)}[Dg]_{\vec a}$

Why? Suppose $z:\mathbb{R}^2\to\mathbb{R}$

$$z\circ\vec u = z\begin{bmatrix}u(x,y)\\ v(x,y)\end{bmatrix} =$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$[D(z\circ\vec u)] = \begin{bmatrix}\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}$$

$$= \frac{\partial z}{\partial \vec u}\cdot\frac{\partial \vec u}{\partial \vec x}$$

$$= [Dz][D\vec u] \longrightarrow$$

Geometrically, this says transform space to $\vec u$, and then z! makes sense!

Generally: $\vec y = \vec y(\vec u)$, $\vec u = \vec u(\vec x)$

$$\boxed{\left.\frac{\partial \vec y}{\partial \vec x}\right|_{\vec x=\vec a} = \left.\frac{\partial \vec y}{\partial \vec u}\right|_{\vec u(\vec a)}\cdot\left.\frac{\partial \vec u}{\partial \vec x}\right|_{\vec x=\vec a}}$$

Ex Suppose $f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin(xyz) \\ e^{xy}+z^2 \\ x+y+z \end{bmatrix}$ and $g\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u^3 \\ u\cos(v) \\ uv \end{bmatrix}$

Compute $[D(f\circ g)]$ at $(u,v,w) = (0,1)$

$$[D(f\circ g)]_{\vec{a}} = [Df]_{g(\vec{a})}[Dg]_{\vec{a}}. \quad g(\vec{a}) = g\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Rmk: When Computing $[D(f\circ g)]_{\vec{a}}$, Plug in numbers BEFORE Multiplying $[Df]$ and $[Dg]$ together! Avoid the mess.

$$[Df]_{(1,0,0)} = \begin{bmatrix} yz\cos(xyz) & xz\cos(xyz) & xy\cos(xyz) \\ ye^{xy} & xe^{xy} & 2z \\ 1 & 1 & 1 \end{bmatrix}_{(1,0,0)}$$

$$[Df]_{(1,0,0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[Dg]_{(0,1)} = \begin{bmatrix} 3u^2 & 0 \\ \cos(v) & -u\sin(v) \\ v & u \end{bmatrix}_{(0,1)} = \begin{bmatrix} 0 & 0 \\ \cos(1) & 0 \\ 1 & 0 \end{bmatrix}$$

$$[D(f\circ g)]_{(0,1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \cos(1) & 0 \\ 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ \cos(1) & 0 \\ 1+\cos(1) & 0 \end{bmatrix}}$$

Ex | If $[D(f\circ g)]_{\vec a} = \begin{bmatrix}1&0\\2&3\end{bmatrix}$ and $[Dg]_{\vec a} = \begin{bmatrix}1&2\\0&2\end{bmatrix}$

What is $[Df]_{g(\vec a)}$?

$$[D(f\circ g)]_{\vec a} = [Df]_{g(a)}\,[Dg]_{\vec a}$$

$$[D(f\circ g)]_{\vec a}\,[Dg]_{\vec a}^{-1} = [Df]_{g(\vec a)}\, I$$

$$[Df]_{g(\vec a)} = \begin{bmatrix}1&0\\2&3\end{bmatrix}\frac{1}{2}\begin{bmatrix}2&-2\\0&1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2&-2\\4&-1\end{bmatrix} = \boxed{\begin{bmatrix}1&-1\\2&-\frac{1}{2}\end{bmatrix}}$$

(Ch 6) Differentiation Rules

Material Derivative:

Here's the idea ⟶ $f=f(t,\vec x)$. $g(t)=\begin{bmatrix}t\\\vec x(t)\end{bmatrix}$

$$f\circ g = f(g(t)) = f\begin{bmatrix}t\\\vec x(t)\end{bmatrix}$$

$$[D(f\circ g)] = [Df][Dg] = \begin{bmatrix}\frac{\partial f}{\partial t} & \frac{\partial f}{\partial \vec x}\end{bmatrix}\begin{bmatrix}1\\\frac{d\vec x}{dt}\end{bmatrix} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec x}\frac{d\vec x}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec x}\vec V.$$

This won't show up. Let's not talk about it.

$P\circ\vec x$

If You do want to talk about Material Derivative:

A particle moves along a curve $\vec{X}(t) = \begin{bmatrix} \sin t = x \\ \sin t \cos t = y \\ \cos^2 t = z \end{bmatrix}$

W/ pressure $\vec{P}(\vec{X},t) = z + x\cos t - y\sin t$

Use the material derivative to compute the ROC of the particle's pressure w/ rspct to time.

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{x}} \frac{d\vec{x}}{dt} \quad \text{for } f(\vec{x},t)$$

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial t} = -x\sin t - y\cos t + \begin{array}{c} \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array} \\ \begin{bmatrix} \cos t & -\sin t & 1 \end{bmatrix} \end{array} \begin{bmatrix} \cos t \\ -\sin^2 t + \cos^2 t \\ -2\cos t \sin t \end{bmatrix}$$

$$\frac{DP}{Dt} = -x\sin t - y\cos t + \cos^2 t + \sin^3 t - \sin t\cos^2 t - 2\cos t\sin t$$

$$= -\sin^2 t - \sin t\cos^2 t + \cos^2 t + \sin^3 t - \sin t\cos^2 t - 2\cos t\sin t$$

$$= \boxed{-\sin^2 t - 2\sin t\cos^2 t + \cos^2 t + \sin^3 t - 2\cos t\sin t}$$

Ex Use chain rule to prove $(u+v)' = u' + v'$

$$f(u,v) = u + v \qquad [D(f \circ g)] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$g(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$$

$$\boxed{[D(f \circ g)] = u' + v'}$$

(Ch7) Inverse Function thrm

Before we get to it, let's ask: What is an inverse? f takes inputs ⟶ outputs
f^{-1} takes outputs back to their respective inputs.

$y = \sin(x)$; $\sin^{-1}(y) = x \rightarrow \sin^{-1}(\sin(x)) = x$

General property of inverse: $f^{-1}(f(x)) = x$

What might an inverse look like for $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$?

Consider this somewhat easy problem:

Ex

$$f\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}2x+2y\\x+3y\end{bmatrix}.$$

$f : (x,y) \longrightarrow (u,v)$. $f^{-1} : (u,v) \longrightarrow (x,y)$

$f(x,y)$ maps to u,v	$f^{-1}(u,v)$ maps to x,y.

Find f^{-1}! This is a function that takes u,v as inputs and spits out the respective x,y.

That is, f^{-1} gives us x and y (outputs) as a function of (u,v): $x(u,v)$
$y(u,v)$.

$$\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}2 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$\vec{u} = A\vec{v}$$

$$A^{-1}\vec{u} = \vec{v}$$

$$\begin{bmatrix}3 & -2\\-1 & 2\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

$$\boxed{\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3u-2v\\-u+2v\end{bmatrix} = f^{-1} = P\begin{bmatrix}u\\v\end{bmatrix}}$$

Check:

$$f^{-1}\left(f\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\y\end{bmatrix}$$

$$P\left(f\begin{bmatrix}x\\y\end{bmatrix}\right) = P\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}x\\y\end{bmatrix} \quad ✓$$

OK, So that what a function inverse might look like w/ matrices.
Let's think about the <u>Inverse Funct. Thrm.</u>

In single variable: $f^{-1}(f(x)) = x$, $\frac{d}{dx}[f^{-1}(f(x))] = \frac{d}{dx}[x]$

$$\frac{d}{dx} f^{-1} \cdot f'(x) = 1 \longrightarrow \boxed{\frac{d}{dx}(f^{-1})\Big|_{f(x)} = \frac{1}{f'(x)|_x}}$$

Same result in higher dimensions: $\boxed{[Df^{-1}]_{f(\vec{x})} = [Df]^{-1}_{\vec{x}}}$

Proof if asked: $(f^{-1}\circ f)(\vec{x}) = \vec{x}$

$$\vec{x}\begin{bmatrix}x_1\\x_2\\\vdots\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\\vdots\end{bmatrix}$$

$$[D(f^{-1}\circ f)] = I$$
$$[Df^{-1}][Df] = I$$
$$[Df^{-1}] = [Df]^{-1}. \quad \blacksquare$$

Geometrically, we established that $[Df]$ shows the local picture of space when going from $(x,y) \longrightarrow f(x,y)$. The inverse of f, by definition, takes us from $f(x,y) \longrightarrow (x,y)$. It is the reverse transformation of space at a particular input. If $[Df]$ shows the local picture of the map $(x,y) \longrightarrow f(x,y)$, $[Df]^{-1}$ shows the local picture of $f(x,y) \xrightarrow{\text{Back to}} (x,y)$. $[Df^{-1}]$ shows the local picture of $f(x,y) \longrightarrow (x,y)$.
They're the same! Cool!

Inverse Funct Thm:

If f is to be locally invertible at $\vec{a}$, the derivative matrix $[Df]_{\vec{a}}$ must be locally invertible. That is: $\det[Df]_{\vec{a}} \neq 0$ if f is invertible near $\vec{a}$.

Why? The derivative $[Df]$ shows you what f does in a region. If $[Df]^{-1}_{\vec{a}}$ DNE, f^{-1} DNE at $\vec{a}$.

Ex $f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x^3 + 3xy \\ 2xy^2 + 3x^2 \end{bmatrix}$. Compute the derivative of the inverse at $f(1,1)$ (if it exists).

$$[Df]_{(1,1)} = \begin{bmatrix} 12x^2 + 3y & 3x \\ 2y^2 + 6x & 4xy \end{bmatrix}_{(1,1)} = \begin{bmatrix} 15 & 3 \\ 8 & 4 \end{bmatrix}. \quad \det = 60 - 24 = 36 \neq 0$$

$$[Df]^{-1}_{(1,1)} = [Df^{-1}]_{f(1,1)} = \boxed{\frac{1}{36}\begin{bmatrix} 4 & -3 \\ 8 & 15 \end{bmatrix}}$$

$\boxed{\text{Ex}}$ Compute the derivative of the inverse of

$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^{x^2} - \cos(y^3) \\ y^4 + 4\sin(2x) \end{bmatrix}$$ at $f(0,0)$ if it exists.

$$[Df]_{\vec{0}} = \begin{bmatrix} 2x \cdot 2e^{x^2} & 3y^2 \sin(y^3) \\ 8\cos(2x) & 3y^3 \end{bmatrix}_{\vec{0}} = \begin{bmatrix} 0 & 0 \\ 8 & 0 \end{bmatrix}$$

$$\det[Df]_{\vec{0}} = 0, \quad \therefore \quad [Df]_0^{-1} = [Df^{-1}]_{f(\vec{0})} \ \boxed{\text{DNE.}}$$

(ch 8) Implicit Function Thrm

Given an implicit function $\vec{F}(\vec{x}, \vec{y}) = \vec{0}$

F : m eqns

$\vec{x}$: n variables

$\vec{y}$: m variables

The Question: Can we express $\overbrace{\vec{y}(\vec{x})}^{\vec{y} \text{ as a funct } \vec{X}}$ near a point $\vec{a}$ which satisfies $\vec{F}(\vec{a}) = \vec{0}$? When can we do this?

$\vec{y}(\vec{x})$ possible when $\det\left[\frac{\partial F}{\partial \vec{y}}\right]_{\vec{a}} \neq 0$

$$\longrightarrow \circledast \quad \left[\frac{\partial \vec{y}}{\partial \vec{x}}\right]_{\vec{a}} = -\left[\frac{\partial F}{\partial \vec{y}}\right]_{\vec{a}}^{-1} \left[\frac{\partial F}{\partial \vec{x}}\right]_{\vec{a}} \qquad \left(\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)$$

How to Use Imp. F.T.

1) Make implicit function F.

2) Ident. $\vec{x}, \vec{y}$

3) If $\det\left[\frac{\partial F}{\partial \vec{y}}\right]_{\vec{a}} \neq 0$: Use ⊛ (above).

Ex Show that near the pt $(x,y,u,v)=(1,1,1,1)$ we can solve

$$xu + yvu^2 = 2$$
$$xu^3 + y^2v^4 = 2$$

Uniquely for u and v as functs. of x, y. Then, find $\left[\frac{\partial \vec{y}}{\partial \vec{x}}\right]_{(1,1,1)}$.

1) Make imp. function:

$$F(x,y,u,v) = \begin{bmatrix} xu+yvu^2-2 \\ xu^3+y^2v^4-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2) Ident. $\vec{y}, \vec{x}$ in therm: $\vec{y} = \begin{bmatrix} u \\ v \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

3) $$\left[\frac{\partial F}{\partial \vec{y}}\right]_{(1,1,1,1)} = \begin{array}{cc|c} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \\ x+2yvu & yu^2 & F_1 \\ 3u^2x & 4y^2v^3 & F_2 \end{array}_{(1,1,1,1)} = \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix}$$

$\det = 9 \neq 0$ ✓ ∴ $\vec{y}(\vec{x})$ is possible near $(1,1,1,1)$

$$\left[\frac{\partial \vec{y}}{\partial \vec{x}}\right]_{\vec{a}} = \left(\left[\frac{\partial F}{\partial \vec{y}}\right]\right)^{-1}_{\vec{a}} \left[\frac{\partial F}{\partial \vec{x}}\right]_{\vec{a}}$$

$$\left[\frac{\partial F}{\partial \vec{x}}\right]_{(1,1,1,1)} = \begin{bmatrix} u & vu^2 \\ u^3 & 2yv^4 \end{bmatrix}_{(1,1,1,1)}$$

$$\left[\frac{\partial F}{\partial \vec{y}}\right]^{-1}_{\vec{a}} = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\left[\frac{\partial \vec{y}}{\partial \vec{x}}\right]_{(1,1,1,1)} = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \boxed{\frac{1}{9}\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}}$$

If Time:

Ex Find eqn of line tangent to unit circle at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$x^2+y^2=1 \rightarrow F(x,y) = x^2+y^2-1 = 0.$$

$$\frac{\partial y}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y} = \text{Slope of line}$$

$$y - y_1 = m(x - x_1)$$

$$m = \left.\frac{\partial y}{\partial x}\right|_{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)} = \left.\frac{-2x}{2y}\right|_{\left(1/2, \sqrt{3}/2\right)} = \frac{-1}{\sqrt{3}}$$

$$\boxed{y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right)}$$

Implicit Function Thrm pf:

Pf: For $F(\vec{x},\vec{y}) = \vec{0}$ where $F: \mathbb{R}^{n+m} \to \mathbb{R}^m$, $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$

use a linear approximation $\vec{z}$ of $F(\vec{x},\vec{y}) = \vec{0}$ near a base point $(\vec{x}_0,\vec{y}_0)$ satisfying $\vec{z}_0 = F(\vec{x}_0,\vec{y}_0) = \vec{0}$.

$\vec{z} = \vec{z}_0 + [DF]_{(\vec{x}_0,\vec{y}_0)} (\vec{x}-\vec{x}_0, \vec{y}-\vec{y}_0)^T$. $z \approx F(\vec{x},\vec{y})$ near $(\vec{x}_0,\vec{y}_0)$ here!

But $F(\vec{x},\vec{y}) = \vec{0}$ → this is the eqn we are trying to solve.

$\vec{z} \sim \vec{F} = 0$

$$0 = 0 + \left[\frac{\partial F}{\partial \vec{x}}\Big|_{(\vec{x}_0,\vec{y}_0)} \;\; \frac{\partial F}{\partial \vec{y}}\Big|_{(\vec{x}_0,\vec{y}_0)} \right] (\vec{x}-\vec{x}_0, \vec{y}-\vec{y}_0)^T$$

$$0 = \overset{n+m \times m}{\left[\frac{\partial F}{\partial \vec{x}}\right]_{(\vec{x}_0,\vec{y}_0)}} \overset{m \times 1}{(\vec{x}-\vec{x}_0)} + \overset{n+m \times n}{\left[\frac{\partial F}{\partial \vec{y}}\right]_{(\vec{x}_0,\vec{y}_0)}} \overset{n \times 1}{(\vec{y}-\vec{y}_0)}$$

Solve for $\vec{y}$: (the goal is to discover something about $\vec{y}(\vec{x})$):

$$\left[\frac{\partial F}{\partial \vec{y}}\right]_{(\vec{x}_0,\vec{y}_0)} (\vec{y}-\vec{y}_0) = -\left[\frac{\partial F}{\partial \vec{x}}\right]_{(\vec{x}_0,\vec{y}_0)} (\vec{x}-\vec{x}_0)$$

$$(\vec{y}-\vec{y}_0) = -\left[\frac{\partial F}{\partial \vec{y}}\right]^{-1}_{(\vec{x}_0,\vec{y}_0)} \left[\frac{\partial F}{\partial \vec{x}}\right]_{(\vec{x}_0,\vec{y}_0)} (\vec{x}-\vec{x}_0)$$

$\vec{x}$ is only variable!

$$\vec{y}(\vec{x}) = \vec{y}_0 - \left[\frac{\partial F}{\partial \vec{y}}\right]^{-1}_{(\vec{x}_0,\vec{y}_0)} \left[\frac{\partial F}{\partial \vec{x}}\right]_{(\vec{x}_0,\vec{y}_0)} (\vec{x}-\vec{x}_0)$$

Suppose $\exists$ a function $g(\vec{x}) = \vec{y}(\vec{x})$ so that $F(\vec{x},\vec{y}) = F(\vec{x}, g(\vec{x})) = \vec{0}$

We expect that $\vec{y}$ is a linear approximation to g (b/c $g(\vec{x}) = \vec{y}$) near $\vec{x}_0$

Linear approximations take on the form $L(X) = L_0 + L'(\vec{X}-\vec{X}_0)$

$$\therefore \; g'(\vec{x}) = -\left[\frac{\partial F}{\partial \vec{y}}\right]^{-1}_{(\vec{x}_0,\vec{y}_0)} \left[\frac{\partial F}{\partial \vec{x}}\right]_{(\vec{x}_0,\vec{y}_0)}$$

w/ $g'(\vec{x}) = \left[\frac{\partial g}{\partial \vec{x}}\right] = \left[\frac{\partial \vec{y}}{\partial \vec{x}}\right]$ iff $\det\left[\frac{\partial F}{\partial \vec{y}}\right]_{(\vec{x}_0,\vec{y}_0)} \neq 0$ (so the inverse can exist).

Geometry!

$$\overset{(m+n)\times m}{\left[\frac{\partial F}{\partial \vec{y}}\right]_{(\vec{x}_0,\vec{y}_0)}} \overset{m\times 1}{(\vec{y}-\vec{y}_0)} = -\overset{(m+n)\times n}{\left[\frac{\partial F}{\partial \vec{x}}\right]_{(\vec{x}_0,\vec{y}_0)}} \overset{n\times 1}{(\vec{x}-\vec{x}_0)}$$

You proved this to be true. $\left[\frac{\partial F}{\partial \vec{y}}\right]$ and $\left[\frac{\partial F}{\partial \vec{x}}\right]$ act as the transformations that collapse $\vec{x}-\vec{x}_0$ and $\vec{y}-\vec{y}_0$ onto each other.

If $\left[\frac{\partial F}{\partial \vec{y}}\right]_{(\vec{x}_0,\vec{y}_0)}$ collapses to null, so will $\left[\frac{\partial F}{\partial \vec{x}}\right]_{(\vec{x}_0,\vec{y}_0)}$. They collapse to the same $(m+n)$ vector. If that vector is null, then $(\vec{y}-\vec{y}_0)$ and $(\vec{x}-\vec{x}_0)$ can be anything and have no dependence on one another.

Prove that the inverse function thrm is a special case of the imp. funct thrm.

If we have $\vec{y}=f(\vec{x})$ for $f:\mathbb{R}^n\to\mathbb{R}^n$. Use the implicit function thrm to show when we may solve for $\vec{x}$ as a funct of $\vec{y}$.

i.e. $\vec{x}=f^{-1}(\vec{y})$

$$\to \vec{y}-f(\vec{x})=\vec{0}$$

$$F(\vec{x},\vec{y})=\vec{y}-f(\vec{x})=\vec{0}$$

Here, we want $\vec{x}=\vec{x}(\vec{y})$. The $\vec{x}$ and $\vec{y}$ in the Imp. FT changed.

$$\det\left[\frac{\partial F}{\partial \vec{x}}\right]\neq 0\ ;\ \det\left[\frac{\partial F}{\partial \vec{x}}\right] = -[Df] \qquad g(\vec{y}) = \vec{x}(\vec{y})$$

$\therefore$ We may express $\vec{x}$ as a funct$(\vec{y})$. Call this function $g(\vec{y})$.

$$\therefore F(\vec{x},\vec{y}) = \vec{y}-f(g(\vec{y})) = 0 \to \vec{y}=f(g(\vec{y}))$$

That is: $g=f^{-1}$

$$D\left[\vec{y} = f(g(\vec{y}))\right];\ [D\vec{y}] = [Df][Dg] = [Df][Df^{-1}]$$

$$[D\vec{y}] = I \quad B/C \quad \vec{y}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}$$

$$I = [Df][Df^{-1}] \rightarrow \underline{[Df]^{-1} = [Df^{-1}]}$$

(ch9) The Gradient. start with this...

Ex $f(x,y,z) = \sin(xy) + e^{yz}$. Find ∇f.

$$\boxed{\nabla f = (y\cos(xy))\hat{i} + (x\cos(xy) + z e^{yz})\hat{j} + (y e^{yz})\hat{k}}$$

for a function $f(\vec{x})$, $\vec{x}$ makes up the input space.

$f(\vec{x})$ gives the output space

4 Key properties of gradient:

1) ∇f gives the direction, in the input space, of greatest function increase.

2) $-\nabla f$ gives the direction, in the input space, of greatest function decrease.

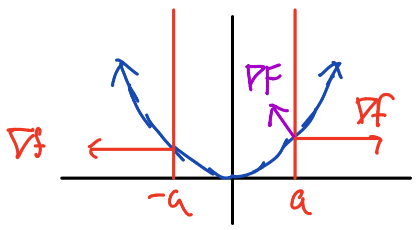
3) For $\nabla f \cdot \vec{u} = 0$, $\vec{u}$ is $\perp$ to ∇f and f does not change in the direction of $\vec{u}$. $\vec{u}$ is tangent to a level set of f.

4) For a surface implicitly defined as $F(x,y,z) = 0$, ∇F is always orthogonal to the surface.

Let's Develop Some intuition: 1D example

$f(x) = x^2$ $\qquad \nabla f = 2x\hat{i}$ $\qquad$ Level sets: $f^{-1}(c) = \{x : f(x) = c\}$

The set of all x for which $f(x) = C$.



For $x > 0$, $\nabla f = +\#\hat{i}$

For $x < 0$, $\nabla f = -\#\hat{i}$

(1,2) • Move ⊥ to level set, f experiences greatest inc. or dec

(note: level set here are the inputs $x = a, -a$. B/c inputs are one dim, moving ⊥ to level set means you can only move in $\pm x$ dir).

(3) • Move up or down level set, f does not change

In General: $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} + \dots$

If $\frac{\partial f}{\partial x}$ is BIG,

- Step in x dir. increases f by large amt.

→ Gradient points us in x dir.

If $\frac{\partial f}{\partial x}$ is small ; $<< 0$

- Step in x dir. decreases f by large amt

→ Gradient points us in $-x$ dir (dir that will increase f).

Conclude:
∇f Points in the direction that will increase f the most. This is always ⊥ to level sets

---

Proof that ∇f is ⊥ to level sets:

Let $\underset{\text{variable}}{w} = f(x,y,z)$. Level set: $f(x,y,z) = C \rightarrow$ constant

Let $\vec{r}(t) = \overbrace{(x(t), y(t), z(t))}^{\text{Level set.}}$ be the set of $\overbrace{x, y, z}^{\vec{x}} \ni f(\vec{x}) = C$.

$$\longrightarrow \quad f(x(t), y(t), z(t)) = C \qquad (\text{Like } x^2+y^2+z^2=4)$$

Let $g(t) = f(x(t), y(t), z(t)) = C$. Recall: $\{x(t), y(t), z(t)\}$ is level set!

Let's evaluate $\frac{dg}{dt}\Big|_{t_0}$. Of course, $g = f = \text{const}$ so the ROC of $g = 0$.

$$\frac{dg}{dt}\Big|_{t_0} = \frac{\partial f}{\partial x}\Big|_{\underset{(x(t_0),\, y(t_0),\, z(t_0))}{\overset{\uparrow}{p}}} \frac{dx}{dt}\Big|_{t_0} + \frac{\partial f}{\partial y}\Big|_p \frac{dy}{dt}\Big|_{t_0} + \frac{\partial f}{\partial z}\Big|_p \frac{dz}{dt}\Big|_{t_0} = 0$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_p \bullet \overbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{\partial z}{\partial z}\right)}^{\text{Vector in plane of level set.}}\Big|_{t_0} = 0$$

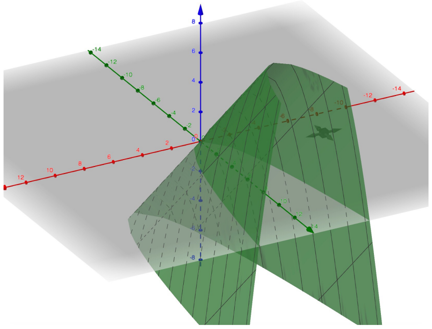
$$\nabla f|_p \cdot \vec{r}\,'|_{t_0} = 0$$

$\nabla f \perp$ Level set for any p, t pair!

Suppose we wanted a vector $\perp$ to the curve $y = x^2$ for any x. That is clearly not ∇f. So what can we do?

We have discussed that the gradient is always $\perp$ to level curves. So: If $y = x^2$ is the level curve of some function, then we can solve this problem easily!

$y = x^2$ is a level curve of what function?



$$\longrightarrow \quad y - x^2 = 0,\ 3, 4, -17 \ldots$$

$$F(x,y) = y - x^2 \ (= 0)$$

$$\nabla F = -2x\hat{i} + \hat{j} \qquad (\nabla F \text{ above})$$

Ex Draw level sets of $f(x,y)=x^2+y^2$.

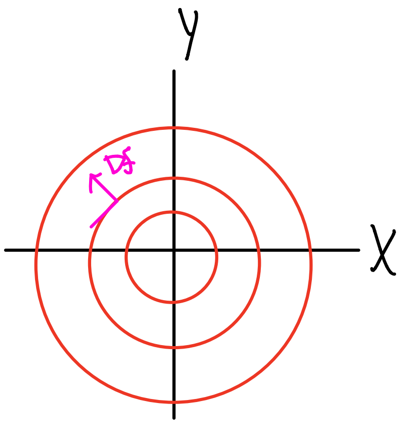
$x^2+y^2=0$

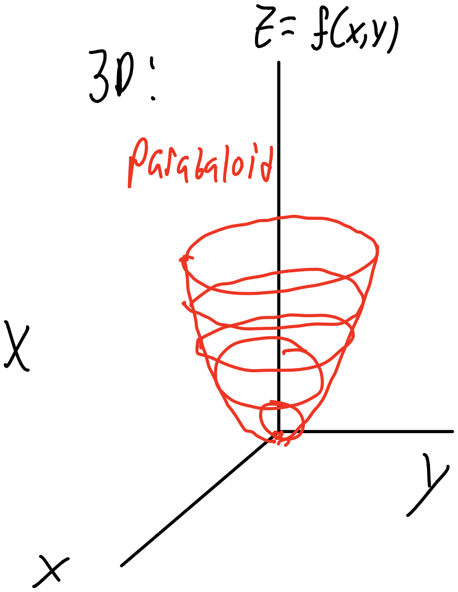
$x^2+y^2=1$

$x^2+y^2=4$

$x^2+y^2=16$

⋮





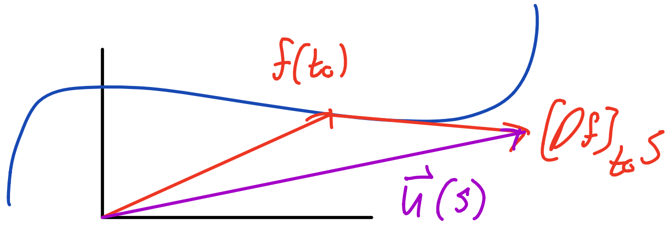
(Ch 10) Tangent Spaces

Parametrized curves

given a parametrized position vector $f(t)=\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix}$, $[Df]=\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{bmatrix}$

$[Df]$ is a velocity vector; always tangent to curve.

∴ A vector tangent to a curve at $f(t_0)$ is $\boxed{\vec{u}(s)=f(t_0)+[Df]_{t_0}s}$



Now,

How can we parametrize a surface?

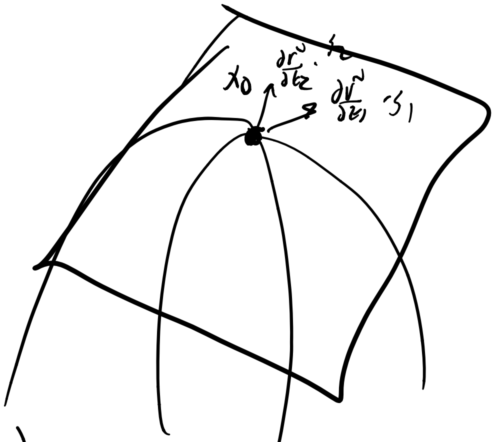
Position vector pointing on surface: $\vec{r}(t_1,t_2)=x(t_1,t_2)\hat{i}+y(t_1,t_2)\hat{j}+z(t_1,t_2)\hat{k}$

$$\frac{\partial \vec{r}}{\partial t_1}=\frac{\partial x}{\partial t_1}\hat{i}+\frac{\partial y}{\partial t_1}\hat{j}+\frac{\partial z}{\partial t_1}\hat{k}$$

$$\frac{\partial \vec{r}}{\partial t_2}=\frac{\partial x}{\partial t_2}\hat{i}+\frac{\partial y}{\partial t_2}\hat{j}+\frac{\partial z}{\partial t_2}\hat{k}$$

So if t_2 is held fixed, $\frac{\partial \vec{r}}{\partial t_1}$ gives us a tangent vector the to surface. Same w/ $\frac{\partial \vec{r}}{\partial t_2}$.

$$\vec{u}(t_1,t_2)=\vec{u}(\vec{t})=\vec{X}_0+\frac{\partial \vec{r}}{\partial t_1}s_1+\frac{\partial \vec{r}}{\partial t_2}s_2$$



$$\vec{u}(s_1,s_2)=\vec{r}_0+\begin{bmatrix}\frac{\partial \vec{r}}{\partial t_1} & \frac{\partial \vec{r}}{\partial t_2}\end{bmatrix}\begin{bmatrix}s_1\\ s_2\end{bmatrix}$$

Christ! $f(t_1,t_2)=\begin{bmatrix}x(t_1,t_2)\\ y(t_1,t_2)\\ z(t_1,t_2)\end{bmatrix}$

$$\boxed{\vec{u}(\vec{s})=f(\vec{t}_0)+[Df]\vec{s}}$$

gives plane tangent to surface!

More importantly, here's what shows up quite often:

For a surface defined by an implicit function $F(\vec{x})=0$, the plane tangent to the surface at $\vec{X}_0$ is the set of all $\vec{x}$ satisfying:

$$\underbrace{\nabla F(\vec{x})\Big|_{\vec{X}_0}}_{\text{Normal to Plane}}\cdot\underbrace{(\vec{x}-\vec{X}_0)}_{\text{Vector in Plane}}=0$$

In $\mathbb{R}^3$: $\nabla F(x,y,z)\Big|_{(x_0,y_0,z_0)} \cdot (x-x_0, y-y_0, z-z_0) = 0$

Ex Find an equation of the plane tangent to surface $x^2+yz+z=3$ at $(x,y,z)=(1,1,1)$:

$F(x,y,z)=x^2+yz+z-3 \; (=0)$

$$\nabla F = 2x\hat{i} + z\hat{j} + (y+1)\hat{k}$$

$$\nabla F\big|_{(1,1,1)} = 2\hat{i}+1\hat{j}+2\hat{k}$$

$$(2,1,1)\cdot(x-1,y-1,z-1)=0$$

$$\boxed{2(x-1)+(y-1)+2(z-1)=0}$$

Ex Give the plane tangent to the parametrized surface

$$\left.\begin{aligned} x(t_1,t_2)&=t_1 \\ y(t_1,t_2)&=t_2 \\ z(t_1,t_2)&=t_1^2+t_2^2+2t_1 \end{aligned}\right\} \quad f\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_1^2+t_2^2+2t_1 \end{bmatrix}$$

@ $t_1=0$, $t_2=1$ as a parametrization w/ parameters s_1, s_2.

$$f(\vec{t}_0) = f\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$[Df]_{(0,1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2t_1+2 & 2t_2 \end{bmatrix}_{(0,1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} x(s_1,s_2) &= 0 + 1s_1 + 0s_2 \\ y(s_1,s_2) &= 1 + 0s_1 + 1s_2 \\ z(s_1,s_2) &= 3 + 2s_1 + 2s_2 \end{aligned}}$$

→ A bit tricky.

Ex | Give two vectors tangent to the parametrized surface in $\mathbb{R}^3$ given by the eq $-4y^2 + 3xy^3 + z^2 = 0$ at $(1,1,1)$. This is a pain to parametrize. So what other tools do we have at our disposal?

We can find the tangent plane at that point and two vectors in the tangent plane!

$$F(x,y,z) = -4y^2 + 3xy^3 + z^2 \qquad (=0)$$

$$\nabla F = 3y^3\,\hat{i} + (-8y + 9xy^2)\,\hat{j} + 2z\,\hat{k}$$

$$\nabla F\Big|_{(1,1,1)} = 3\hat{i} + \hat{j} + 2\hat{k}$$

$$3(x-1) + (y-1) + 2(z-1) = 0$$

$$3x - 3 + y - 1 + 2z - 2 = 0$$

$$\underline{3x + y + 2z = 6}$$

we need 3 points in the plane to deduce 2 tangent vectors.

$A(1,1,1)$

chose $x=1, y=2$

$3+2+2z=6$

$2z=1$

$z=1/2$

$B(1,2,1/2)$

Choose $x=2, y=1$

$6+1+2z=6$

$2z=-1$

$z=-1/2$

$C(2,1,-1/2)$

$$\vec{AB}=(1-1,2-1,\tfrac{1}{2}-1)=\boxed{(0,1,-1/2)}$$

$$\vec{AC}=(2-1,1-1,-1/2-1)=\boxed{(1,0,-3/2)}$$

Check your answer by taking $\vec{AB}\times\vec{AC}$. It should be some multiple of $\nabla F|_{(1,1,1)}$.

(Ch 11) Linearization

This is important...

Given a function $f(x,y,z)$, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

Like: $dy = f'(x)\,dx \to \Delta y \approx f'(x)\Delta x$

$$\longrightarrow \Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$$

Relative error $= \frac{\Delta f}{f} \approx \frac{df}{f}$

Ex A cylindrical solid can is designed to have radius r and height h. If measurement in h can vary by up to 10%, r up to 20%, by what percent can volume of the can vary?

Relative error: $|\Delta h| = 0.10h$, $|\Delta r| = 0.20r$, $|\Delta V| \leq ??\,V$

Remember: $\Delta h \approx dh$, $\Delta r \approx dr$, $\Delta V \approx dV$.

$$\Downarrow$$

$\frac{|\Delta V|}{V} \leq ??$

$$V(r,h) = \pi r^2 h$$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$$

$$\Delta V \approx \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h$$

$$\frac{\Delta V}{V} \approx \frac{\partial V}{\partial r}\frac{\Delta r}{V} + \frac{\partial V}{\partial h}\frac{\Delta h}{V}$$

We are looking from $\max\left(\frac{|\Delta V|}{V}\right)$ so that $\frac{|\Delta V|}{V} \leq \#$ always. The max error in volume!

So here's the question: How can we maximize the RHS so that no $\frac{|\Delta V|}{V}$ can be bigger?
Take that abs. value of each term that can be negative.

$$\frac{|\Delta V|}{V} \le \left|\frac{\partial V}{\partial r}\right| \frac{|\Delta r|}{V} + \left|\frac{\partial V}{\partial h}\right| \frac{|\Delta h|}{V}$$

$$= \frac{2\cancel{\pi}\cancel{r}\cancel{h}}{\cancel{\pi}r^2\cancel{h}}|\Delta r| + \frac{\cancel{\pi}\cancel{r^2}}{\cancel{\pi}\cancel{r^2}h}|\Delta h|$$

$$= 2\frac{|\Delta r|}{r} + \frac{|\Delta h|}{h}$$

$$= 2(0.20) + 0.10$$

$$= 0.50$$

$$\frac{|\Delta V|}{V} \le 0.50$$

Volume measurement can vary by up to 50%

Ex | This hasn't shown up but I will include it anyhow.

Approximate the value of $\ln(e+0.02)\frac{1}{0.49}$ using differentials.
Express your answer in a fraction, not decimal.

Idea: $f(x,y) = \frac{\ln(e+x)}{y}$

$$f(x+\Delta x, y+\Delta y) = f(x,y) + df$$

Let $x=0$, $y=0.50$

$\Delta x = 0.02$, $\Delta y = -0.01$

$$df = \frac{1}{y}\,\frac{1}{e+x}\,dx - \frac{1}{y^2}\ln(e+x)\,dy$$

$$df\Big|_{\substack{dx=0.02\\ dy=-0.01\\ x=0\\ y=0.5}} = 2\,\frac{1}{e}(0.02) - 4\ln(e)(-0.01)$$

$$= 2\,\frac{1}{e}\,\frac{2}{100} + 4\,\frac{1}{100}$$

$$df = \frac{4}{100}\left[\frac{1}{e}+1\right] = \frac{1}{25}\left[\frac{1}{e}+1\right]$$

$$\therefore f(0.02, 0.49) = f(0, 0.5) + df$$

$$\approx 2\ln(e) + df = \boxed{2 + \frac{1}{25}\left[\frac{1}{e}+1\right]}$$

$$\approx 2.054$$

Actual $= 2.055$.

(Ch 12/13) Multivariate Taylor Series

There's a crazy formula.... where did it come from

Intuition:

Taylor series in 1-variable, we started by assuming that $f(x)$ has a power series representation

$$T(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

$$\left.\begin{aligned} f(a) &= T(a) \\ f'(a) &= T'(a) \\ f''(a) &= T''(a) \end{aligned}\right\} \text{This gave the appropriate constants}$$

$$\vdots$$

We can do the same in many variables:

$$T(x,y) = c_0 + c_1(x-a) + c_2(y-b) + c_3(x-a)(y-b) + \dots$$

$$f(a,b) = T(a,b)$$

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial T}{\partial x}(a,b)$$

$$\vdots$$

Leads to a pattern captured by this formula:

About $\vec{x} = \vec{a}$

$$\boxed{f(\vec{x}) = \sum_{I} \frac{1}{I!} D^{I} f\Big|_{\vec{a}} (\vec{x} - \vec{a})^{I}}$$

Properties of mult-index: purely definitional... and useful!

Powers

$$\vec{X} = (u,v,w), \quad I = (1,2,3)$$

$$\vec{X}^I = \vec{X}^{(1,2,3)} = (u,v,w)^{(1,2,3)} = uv^2w^3$$

Derivatives

$$f(x,y,z) = x^2 + y^3 z, \quad I = (2,2,1)$$

$$D^I f = D^{(2,2,1)} f = \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial z} f = \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial z}(x^2+y^3z) = 0$$

$$I = (0,0,0)$$

$$D^I f = f = x^2 + y^3 z.$$

Factorials

$$I = (i_1, i_2, i_3, \ldots, i_n)$$

$$I! = i_1!\, i_2!\, i_3! \ldots i_n! \qquad |I| = i_1 + i_2 + \ldots + i_n$$

$$I = (3,2,1)$$

$$I! = 3!\,2!\,1! = 12$$

$$I = (0,3,0)$$

$$I! = 0!\,3!\,0! = 6$$

Summation

$\sum_I A$. Suppose $I = (i_1, i_2)$ (2 elements)

This says sum over all possible multi-indices

$$\sum_I A = A_{(0,0)} + A_{(1,0)} + A_{(0,1)} + A_{(1,1)} + A_{(2,1)} + \ldots$$

Ex Compute the taylor series $f(x,y) = 2 - x + 2y + 4xy + x^2$ about $(1,3)$.

This is a deg. 2 polynomial!

$$\sum_{I=(i_1,i_2)} \underbrace{\frac{1}{I!} D^{I} f\Big|_{(a,b)} (x-a, y-b)^{(i_1,i_2)}}_{A} = A_{(0,0)} + A_{(1,0)} + A_{(0,1)} + A_{(1,1)} + A_{(2,0)} + A_{(0,2)}$$

Make a table:

A	$I!$	D^I	@ $(1,3)$	$(x-1)^{i_1}(y-3)^{i_2}$
$A_{(0,0)}$	1	f	20	1
$A_{(1,0)}$	1	$-1+4y+2x$	13	$(x-1)$
$A_{(0,1)}$	1	$2+4x$	6	$(y-3)$
$A_{(1,1)}$	1	4	4	$(x-1)(y-3)$
$A_{(2,0)}$	2	2	2	$(x-1)^2$
$A_{(0,2)}$	2	0	0	$(y-3)^2$

$$\begin{aligned} f(x,y) &= 20 + 13(x-1) + 6(y-3) + 4(x-1)(y-3) + \tfrac{1}{2}\cdot 2(x-1)^2 \\ &= 20 + 13x - 13 + 6y - 18 + 4(xy - 3x - y + 3) + x^2 - 2x + 1 \\ &= -10 + 11x + 6y + 4xy - 12x - 4y + 12 + x^2 \\ &= 2 - x + 2y + 4xy + x^2 \end{aligned}$$

✓ Recovered original f.

Let's talk the Substitution method...

Ex Compute the taylor series of $\ln(1+\sin(xy))+e^{xz}$ about the origin up to order 4.

$$\ln(1+u)=u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}$$
$$\sin(z)=z-\frac{z^3}{3!}+\frac{z^5}{5!}$$
$$e^p=1+p+\frac{p^2}{2!}+\frac{p^3}{3!}+\frac{p^4}{4!}$$

Just to include it:

$$\cos(q)=1-\frac{q^2}{2!}+\frac{q^4}{4!}-\frac{q^6}{6!}+\dots$$

$$\ln(1+\sin(z))=\sin(z)-\frac{(\sin(z))^2}{2}+\frac{(\sin(z))^3}{3}$$
$$=z-\frac{z^3}{3!}-\frac{1}{2}\left(z-\frac{z^3}{3!}\right)^2+\frac{1}{3}\left(z-\frac{z^3}{3!}\right)^3$$
$$=xy-\cancel{\frac{(xy)^3}{3!}}-\frac{1}{2}\left(xy-\cancel{\frac{(xy)^3}{3!}}\right)^2+\frac{1}{3}\left(\cancel{xy}-\cancel{\frac{(xy)^3}{3!}}\right)^3$$

$$\rightarrow \ln(1+\sin(xy))=xy-\frac{1}{2}(xy)^2$$

$$e^{xz}=1+xz+\frac{(xz)^2}{2!}+\cancel{\frac{(xz)^3}{3!}}$$

$$\longrightarrow \boxed{xy-\frac{1}{2}(xy)^2+1+xz+\frac{(xz)^2}{2}}$$

Question to consider: What is happening at the origin? You will be able to tell soon!

We begin to answer this question by defining the Hessian

Hessian: Matrix of 2nd Partial Derivatives, different for $[Df]$.

For $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^1$

$$[D^2 f] = \begin{array}{c|cc} & f_x & f_y \\ \hline \frac{\partial}{\partial x} & \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial}{\partial y} & \frac{\partial f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{array}$$

(Ch 14) Critical Points and Optimization

How to Classify a Critical point

$[D^2 f]$

$\det[D^2 f]_{\vec{a}} < 0$ SADDLE

$\det[D^2 f]_{\vec{a}} > 0$

$\longrightarrow \mathrm{Tr} > 0$ Local Min

$\longrightarrow \mathrm{Tr} < 0$ Local Max

Hessian intuition

$$[D^2f] = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \; ; \; H = \det[D^2f] = f_{xx}f_{yy} - (f_{xy})^2$$

If $H|_{\vec{a}} > 0$, f_{xx} f_{yy} could be

$+$ $+$ local min

$-$ $-$ local max

$Tr(D^2f) = f_{xx} + f_{yy}$

>0, $\underset{+}{f_{xx}} + \underset{+}{f_{yy}} > 0$ local min case

<0, $\underset{-}{f_{xx}} + \underset{-}{f_{yy}} < 0$ local max case

If f_{xx}, f_{yy} have opposite signs, $H<0$. SADDLE.

Ex Compute and classify critical points of $2x^2+2y-4y^2+4x$

$f(x,y) = 2x^2+2y-4y^2+4x$

$\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $\nabla f = 0$ $([Df] = [0 \cdots 0]$

$$[Df] = [4x+4 \quad 2-8y] = [0 \quad 0]$$

$x+1=0 \qquad 1-4y=0$

$x=-1 \qquad y = \frac{1}{4} \longrightarrow (-1,\frac{1}{4})$

Classify: $[D^2f] = \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix}$; $\det < 0 \longrightarrow$ $\boxed{(-1,\frac{1}{4}) \text{ is a SADDLE Pt.}}$

Ex Classify all Critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$

$$[Df] = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$3x^2 = 3y \qquad 3y^2 = 3x$$

$$x^2 = y \qquad y^2 = x$$

$\underline{x=0}, \underline{y=0}$ is a clear choice.

Investigate further:

$$(x^2)^2 = x$$

$$x^4 = x$$

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0 \qquad x = 0, \ \underline{x=1} \to \underline{y=1}$$

Two crits: $(0,0)$, $(1,1)$

$$[D^2 f] = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix} \longrightarrow \det[D^2 f] = 36xy - 9$$

$\det[D^2 f]_{(0,0)} = -9 < 0$. $\therefore$ SADDLE

$\det[D^2 f]_{(1,1)} = 27 > 0$ $\therefore$ Local Min

$\operatorname{tr} > 0$

$(0,0)$: Saddle

$(1,1)$: Local min

(Ch 16) Nash Equillibrium

For example, imagine a game between Tom and Sam. In this simple game, both players can choose strategy A, to receive $1, or strategy B, to lose $1. Logically, both players choose strategy A and receive a payoff of $1. If you revealed Sam's strategy to Tom and vice versa, you see that no player deviates from the original choice. Knowing the other player's move means little and doesn't change either player's behavior. The outcome A, A represents a Nash Equilibrium.

Nash equilibrium is the most optimal compromise.
To find that optimum, use payoff matrices.

Payoff matrix: P. P is always relative to player A.

Ex | In the even odd game there are two players. Each extends 1 or 2 fingers.
If the sum total = odd, odd wins that amt. Else, even wins.
What is the payoff matrix that describes this game?

Player A: Row strategies

Player B: Col strategies

$$P = \begin{array}{c|cc|l} A_1 & -2 & 3 & \text{(Player A Strat. 1)} \\ A_2 & 3 & -4 & \text{(Player A Strat. 2)} \\ \hline & B_1 & B_2 & \\ & \text{(Player B Strat 1.)} & \text{(Player B Strat 2.)} & \end{array}$$

Average Expected Payout from A ⟶ B.

What would happen if > 100 rounds are played. What would be the average net gain or loss for Player A? (of course this tells us about avg net loss or gain for B).

This single # (avg expected payoff) depends on how frequently each player chooses a strategy. Let $\vec{a}$ be a column vector whose first row is how frequently A chooses strategy 1, 2nd row shows freq of second strategy.

Let there be a column vector of the same structure for B (call this vector $\vec{b}$).

$$\vec{a} = \begin{bmatrix} a \\ 1-a \end{bmatrix} \begin{array}{l} \leftarrow \text{freq. A Chooses Strat 1.} \\ \leftarrow \text{freq A chooses Strat 2.} \end{array}$$

$$\vec{b} = \begin{bmatrix} b \\ 1-b \end{bmatrix}$$

$$\left. \begin{array}{l} a+1-a=1 \\ b+1-b=1 \end{array} \right\} 0 \leq (a,b) \leq 1$$

Clearly, the avg expected payoff from A → B depends on the frequency that strategies are chosen. If f = avg payoff A → B, $f = f(a,b)$.

$f(a,b)$ is a weighted average. Again: Avg payoff from A → B. (using even odd game).

(Bigger Payoff) ^

$$f(a,b) = \underbrace{\underbrace{(a)(b)}_{\text{frequency}}\underbrace{(-2)}_{\text{weight}}}_{\substack{\text{1st row strat.} \\ \text{1st col strat.}}} + \underbrace{(a)(1-b)(3)}_{\substack{\text{1st row strat.} \\ \text{2nd col strat.}}} + (1-a)(b)(3) + (1-a)(1-b)(-4)$$

$$= \vec{a}^T P \vec{b}$$

$$= \overset{1\times 2}{\begin{bmatrix} a & 1-a \end{bmatrix}} \overset{2\times 2}{\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}} \overset{2\times 1}{\begin{bmatrix} b \\ 1-b \end{bmatrix}} = \overset{1\times 2}{\left[(-2)(a)+(3)(1-a)\ ,\ (3)(a)+(-4)(1-a)\right]} \overset{2\times 1}{\begin{bmatrix} b \\ 1-b \end{bmatrix}}$$

$$f(a,b) = \left[(-2)(a)(b) + (3)(1-a)(b) + (3)(a)(1-b) + (-4)(1-a)(1-b)\right]$$

$$\rightarrow \boxed{f(a,b) = \vec{a}^T P \vec{b}}$$

The rest of the story is seeing if an optimal condition exists.
Let's analyze a slightly more complicated game: Mendelshon game.

$$P = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

Compute the Nash eq. for the payoff matrix and probability vectors.

$$\vec{a} = \begin{bmatrix} a \\ b \\ 1-a-b \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} c \\ d \\ 1-c-d \end{bmatrix}$$

What is the payoff at equilibrium?

① Compute avg. expected payoff : $f(a,b,c,d) = \vec{a}^T P \vec{b}$

② $\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} = \frac{\partial f}{\partial d} = 0$

$\rightarrow a_0 \rightarrow b_0 \rightarrow c_0 \rightarrow d_0$

③ Calculate $f(a_0, b_0, c_0, d)$.

① $f(a,b,c,d) = \vec{a}^T P \vec{b} = 4ad - 2a + b - 4bc + 2c - d$

② $\frac{\partial f}{\partial a} = 4d - 2 = 0 \rightarrow d = \frac{1}{2}$

$$\frac{\partial f}{\partial b} = 1-4c = 0 \rightarrow c = \frac{1}{4}$$

$$\frac{\partial f}{\partial c} = -4b+2 = 0 \rightarrow b = \frac{1}{2}$$

$$\frac{\partial f}{\partial d} = 4a-1 = 0 \rightarrow a = \frac{1}{4}$$

At equilibrium: $\vec{a} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$

③ $\boxed{f(1/4, 1/2, 1/4, 1/2) = 0}$

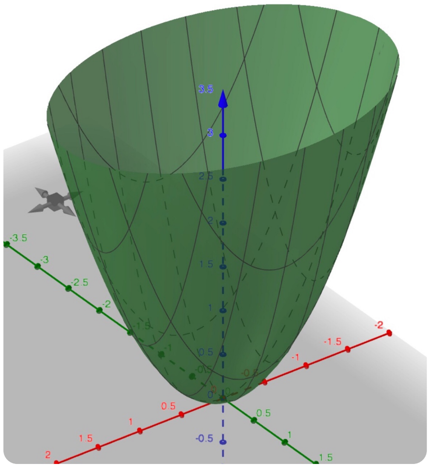
Avg payoff from A→B at Nash Eq. is best payoff both players can simultaneously attain.
Here, the most optimal compromise is 0. A nor B experience any net gain or loss. No player has advantage. It's all give and take.

(Ch 17) Constrained Optimization

We already saw that we have a critical point when $[Df] = \vec{0}$.
What happens if we constrain the domain of the function we are trying to optimize?

Think visually! If we have a function $f(x,y)=z=2y^2+x^2$

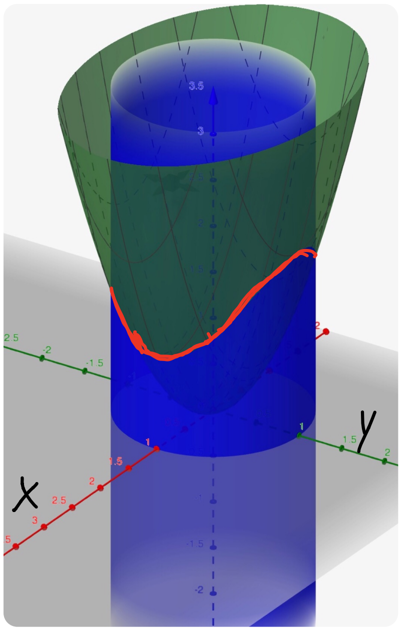
This is an elliptic paraboloid. $x=0$, $z=2y^2$, $y=0$ $z=x^2$



Clearly, this has no critical points other than $x=y=0$

But what if we constrain the domain to $D:\{(x,y): x^2+y^2\le 1\}$

We get this:



Blue: $x^2+y^2=1$
Green: $z=2y^2+1$

$\}$ All points inside the cylinder or on the bdry can qualify as critical points. Anything outside cylinder is no good.

Let's check for crits (even though it's obvious):

$$[Df]=[2x \quad 4y]=[0 \quad 0] \rightarrow (x,y)=(0,0)$$

$\uparrow$ minimum

But we must have 4 critical points on the bdry!

How to find? Constrain f to the red boundary. If we can think of a function that behaves like f but only takes inputs on the red bdry, we can treat the problem like any old 1-d calculus optimization question.

We can!

$x^2+y^2 \leq 1 \rightarrow$ Check $x^2+y^2=1$

$$0 \leq t \leq 2\pi \begin{cases} x = \cos t \\ y = \sin t \end{cases} f(x(t), y(t)) = 2\sin^2 t + \cos^2 t = f(t)$$

$$\frac{\partial f}{\partial t} = 4\sin t\cos t - 2\cos t\sin t = 0$$

$$2\sin t\cos t = 0$$

$\sin t = 0 \qquad \cos t = 0$

$t = 0 \qquad t = \pi/2$

$t = \pi \qquad t = \frac{3\pi}{2}$

$f(0) = 1 \qquad f(\pi/2) = 2$

$f(\pi) = 1 \qquad f(3\pi/2) = 2$ $\Big\}$ maximum value!!

$\boxed{\therefore 0 \text{ is min value of } f,\ 2 \text{ is max (on } D)}$

(Ch 18/19) Lagrange Multipliers

"Constraint problems get messy. What's another way to do them compared to before?" (plugging constraint into f).

Use Lagrange Multipliers

How it works: Let's work in 2D to build a picture

$f(x,y)$: function you want to optimize

$G(x,y)=0$: Constraint

Here's what we know. $G(x,y)=0$ is a specific level set of G. It is the set of all $x,y \ni G(x,y)=0$. $\therefore \nabla G$ is $\perp$ to $G(x,y)=0$ (∇G is $\perp$ to level set of G).

Side Note

λ is the RoC of the optimal value w/ rspct to the constraint value: $G(\vec{x})=c$; $[\nabla f]=\lambda[\nabla G]$

$$G-c=0$$

$$\vec{x}=\vec{x}(c) \rightarrow \text{set of all } \vec{x}$$

$$\ni G(\vec{x})=c$$

$$\frac{\partial f}{\partial c}=\frac{\partial f}{\partial \vec{x}}\frac{\partial \vec{x}}{\partial c}$$

$$=\lambda\frac{\partial G}{\partial \vec{x}}\frac{\partial \vec{x}}{\partial c}$$

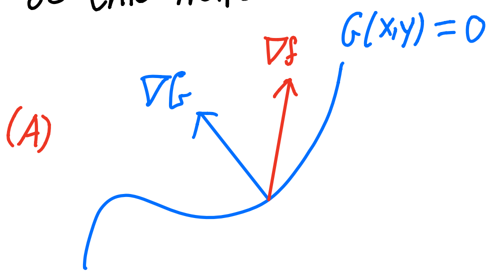
$$=\lambda\frac{\partial G}{\partial c}\quad(G=c)$$

$$\frac{\partial f}{\partial c}=\lambda\frac{\partial c}{\partial c}=\lambda$$

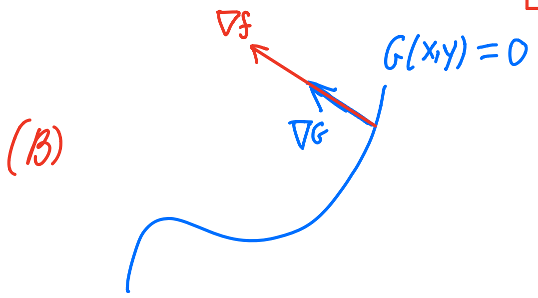
$$\frac{\partial}{\partial c}f(x(c))=\lambda$$

So this helps

(A)



(B)



- ∇G is $\perp$ to $G=0$
- f takes inputs along the path $G=0$
- ∇f gives direction to move in (x,y) plane to increase f the most.
- Motion $\perp$ to ∇f means f will not change
- If ∇f is not in line w/ ∇G, moving along G will cause f to change. (A) (motion has comp. in dir of ∇f).
- If ∇f and ∇G are in line, motion along G moves you $\perp$ to ∇f (B)
  $\therefore f$ is at a critical point (No inc. or dec. in f in this vicinity).

EX Find the least distance between the origin and the plane

$X+3y-2z=4$

① Ident. function (trying to optimize)

② Ident. Constraint

① function: $\text{dist} = \sqrt{x^2+y^2+z^2}$

But optimizing (dist) = optimizing (dist^2)

$\longrightarrow f(x,y,z) = x^2+y^2+z^2$

② Constraint: All points that we may feed to f lay on the plane

$X+3y-2z=4 \longrightarrow$ Def: $G(x,y,z) = X+3y-2z-4 \ (=0)$

$[Df] = \lambda[DG]$: Find x,y,z for which this holds

$$[2x \quad 2y \quad 2z] = \lambda[1 \quad 3 \quad -2]$$

$2x=\lambda \longrightarrow x=\frac{\lambda}{2}$

$2y=3\lambda \longrightarrow y=\frac{3\lambda}{2}$

$2z=-2\lambda \longrightarrow z=-\lambda$

Plug into Constraint:

$\frac{\lambda}{2}+3\left(\frac{3}{2}\lambda\right)-2(-\lambda)-4=0$

$\frac{\lambda}{2}+\frac{9}{2}\lambda+2\lambda-4=0$

$7\lambda=4$

$\lambda=\frac{4}{7}$

$X=\frac{4}{14}=\frac{2}{7}$

$y=\frac{3}{2}\cdot\frac{4}{7}=\frac{6}{7}$

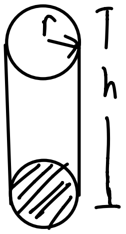
$z=-\frac{4}{7}$

Greatest dist from origin to any plane $\longrightarrow \infty$

∴ This is a min.

$$\boxed{\text{dist}_{min} = \left(\left(\frac{2}{7}\right)^2+\left(\frac{6}{7}\right)^2+\left(\frac{4}{7}\right)^2\right)^{1/2} = \frac{2\sqrt{14}}{7}}$$

Ex Maximize the Volume of an open top cylindrical can of radius r, height h, fixed surface area α.



$V(r,h) = \pi r^2 h$ function

$\alpha = 2\pi r h + \pi r^2$ Constraint

$\longrightarrow S(r,h) = 2\pi r h + \pi r^2 - \alpha \; (=0)$

$$[DV] = \lambda [DS]$$

$$[2\pi r h \quad \pi r^2] = \lambda [2\pi h + 2\pi r \quad 2\pi r]$$

① $\cancel{2\pi} r h = \cancel{2\pi} h \lambda + \cancel{2\pi} r \lambda$

② $\cancel{\pi} r^{\cancel{2}} = 2\cancel{\pi}\cancel{r}\lambda$

① $rh = \lambda(h+r)$ ② $r = 2\lambda$

$\lambda = \dfrac{rh}{h+r}$ $\lambda = \frac{1}{2} r$

$$\lambda = \lambda$$

$$\frac{\cancel{r}h}{h+r} = \frac{1}{2}\cancel{r}$$

$\dfrac{2h}{h+r} = 1 \longrightarrow 2h = h + r \longrightarrow h = r$ (This is the h, r that satisfies 1, 2).

Looking for max Vol for fixed SA α:

$\longrightarrow$ Constraint: $\alpha = 2\pi h^2 + \pi h^2 = 3\pi h^2$

$$\boxed{h = r = \sqrt{\frac{\alpha}{3\pi}}}$$

Ex Find all critical points of $f(x,y)=2x^3+\frac{4}{3}y^3$ constrained to a circle of radius 2. Use lagrange multipliers.

$f(x,y)=2x^3+\frac{4}{3}y^3$ (funct)

(constraint) $x^2+y^2=4 \to G(x,y)=x^2+y^2-4 \; (=0)$

$$[Df]=\lambda[DG]$$

$$\begin{bmatrix}6x^2 & 4y^2\end{bmatrix}=\lambda\begin{bmatrix}2x & 2y\end{bmatrix}$$

$6x^2=\lambda 2x \to 3x^2=\lambda x$ ①

$4y^2=\lambda 2y \to 2y^2=\lambda y$ ②

Find the x,y for which these eqns hold. These x,y will be extreme values!

① $3x^2=\lambda x$

For $x=0$

$0=0$ (① satisfied)

$y=??$

Plug $x=0$ into constraint to find out

$y^2=4 \to y=\pm 2 \quad (2(\pm2)^2=\lambda(\pm2))$

$\boxed{(0,2),\ (0,-2)}$

Are we done? NO! What if $x\neq 0$??

For $x\neq 0$

① $3x^2=\lambda x \to 3x=\lambda$

② $2y^2=\lambda y$

$2y^2=(3x)y$ $(x\neq 0)$

For $y=0$, $0=0$ satisfies ②

"What's" $x=??$

$x^2=4$ (constraint)

$x=\pm 2$

$\boxed{(2,0),\ (-2,0)}$

Are we done yet? No!!

Rmrk: If you can, eliminate λ to solve for x,y directly

What is $y \neq 0$??

Had: $2y^2 = (3x)y$ ($x \neq 0$)

For $y \neq 0$,

$2y = 3x$

$y = \frac{3}{2}x$

$\rightarrow x^2 + \frac{9}{4}x^2 = 4$ (constraint)

$\frac{13}{4}x^2 = 4$

$x^2 = \frac{16}{13}$

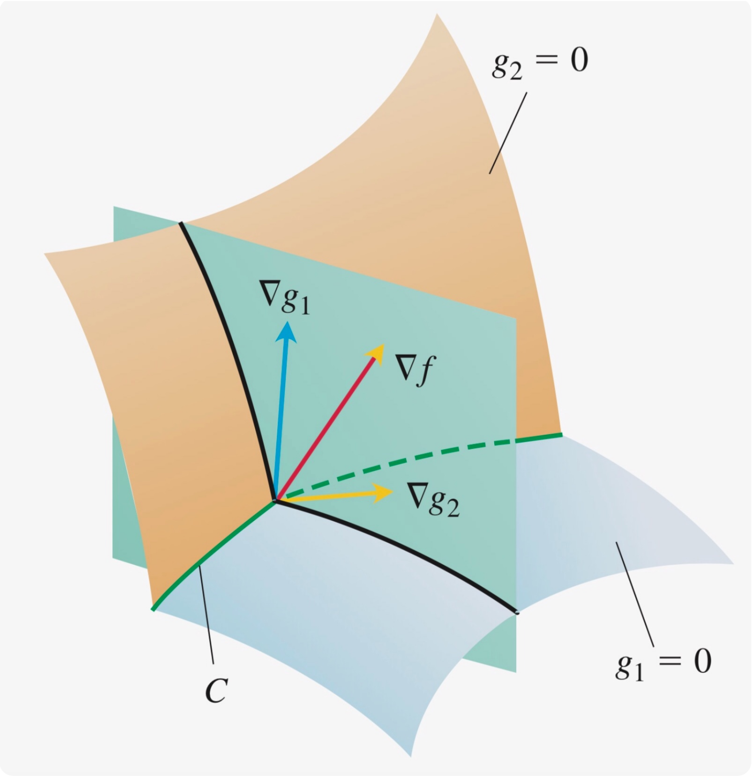
$x_1 = \frac{4}{\sqrt{13}}$, $x_2 = \frac{-4}{\sqrt{13}}$

$y_1 = \frac{3}{2}\left(\frac{4}{\sqrt{13}}\right) = \frac{6}{\sqrt{13}}$, $y_2 = \frac{-6}{\sqrt{13}}$

$$\boxed{\left(\frac{4}{\sqrt{13}}, \frac{6}{\sqrt{13}}\right), \left(\frac{-4}{\sqrt{13}}, \frac{-6}{\sqrt{13}}\right)}$$

Extra: Double Lagrange

$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$; g_1, g_2 are constraints



∇f in same plane as $\nabla G_1, \nabla G_2$

Next page for example.

Ex| **Double Lagrange** (Extra)

The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.

$\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2$; G_1, G_2 are constraints

Show picture: ∇f in same plane as $\nabla G_1, \nabla G_2$

$(\text{distance})^2$: $f(x,y,z) = x^2+y^2+z^2$

$G_1(x,y,z) = x^2+y^2-1 = 0$

$G_2(x,y,z) = x+y+z-1 = 0$

$$2x\hat{i}+2y\hat{j}+2z\hat{k} = \lambda_1(2x\hat{i}+2y\hat{j})+\lambda_2(\hat{i}+\hat{j}+\hat{k})$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k} = (2\lambda_1 x+\lambda_2)\hat{i}+(2\lambda_1 y+\lambda_2)\hat{j}+\lambda_2\hat{k}$$

$2x = 2\lambda_1 x+\lambda_2$ $\quad$ $2y = 2\lambda_1 y+\lambda_2$ $\quad$ $\underline{2z = \lambda_2}$

$\cancel{2}x = \cancel{2}\lambda_1 x + 2z \implies (1-\lambda_1)x = z$ ①

$2y = 2\lambda_1 y + \cancel{2}z \implies (1-\lambda_1)y = z$ ②

①, ② are satisfied if either $\lambda_1 = 1$, $z=0$, or $\lambda \neq 1$ for which:

$$x = y = \frac{z}{1-\lambda_1}$$

For $z=0$: Use G_1 and G_2 functions to find appropriate x, y.

$x^2+y^2=1$

$x+y=1 \to y=1-x$

$x^2+\cancel{1}-2x+x^2=\cancel{1} \to 2x^2-2x=0 \longrightarrow$

$x(x-1)=0$

$x=0,\ x=1,\ z=0;\ y=1-x$

$\boxed{((0,1,0),\ (1,0,0)}$

we had $x=y=\frac{z}{1-x_1}$

Use G_1, G_2 again:

$G_1: x^2+y^2-1=0$ $\qquad$ $G_2: x+y+z-1=0$

$2x^2=1$ $\qquad$ $2x+z-1=0$

$x_1=+\frac{1}{\sqrt{2}}\quad x_2=\frac{-1}{\sqrt{2}}$ $\qquad$ $z=1-2x$

$\longrightarrow z_1=1-2\left(\frac{1}{\sqrt{2}}\right)=1-\sqrt{2}$

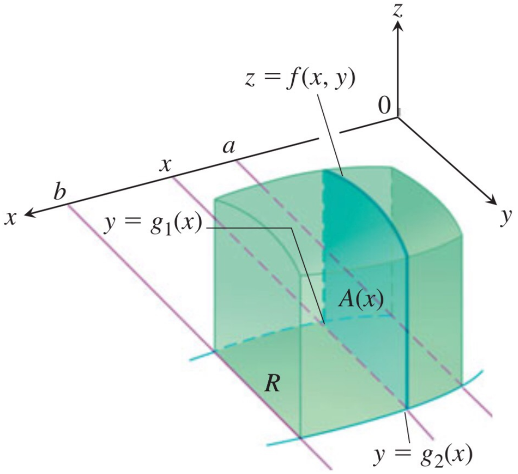
$z_2=1+2\left(\frac{1}{\sqrt{2}}\right)=1+\sqrt{2}$

$$\boxed{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1-\sqrt{2}\right),\ \left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}},1+\sqrt{2}\right)}$$

Vol 3

(Ch 1-3) Double Integrals

Start With Intuition:



We have a block type object. We want to find the volume of the block.

The block has height $Z = f(x,y)$

Question: What is the area of cross section at X?

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} \overset{\swarrow \text{height}}{f(x,y)} \underset{\nwarrow \text{base}}{dy}$$ (Easy math 104 problem)

Now, add all of the cross sections up to get a volume

That's how triple integrals work!

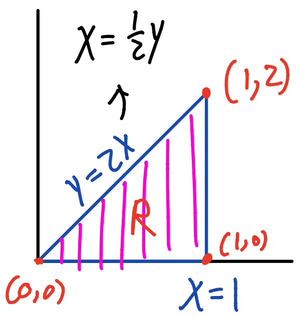
$$V = \int_{x=a}^{x=b} A(x)\,dx = \int_{x=a}^{x=b} \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy \right) dx = \overset{\text{step further}}{} \int_{x=a}^{b} \int_{y=g_1}^{g_2} \int_{z=0}^{f(x,y)} dz\,dy\,dx$$

Ex Integrate ye^{x^3} over the triangular region bounded by $y=2x$, the x axis, line $x=1$.

① Sketch → Label Corners. Do this on exams!

② Consider Limits

③ Integrate



Let's start by integrating w/ rspct to x first:

$$\tfrac{1}{2}y \le x \le 1$$

$$0 \le y \le 2$$

$$\longrightarrow \iint_R f\,dA = \int_{y=0}^{2}\Big(\underbrace{\int_{x=\frac{1}{2}y}^{1} ye^{x^3}dx}_{A(x)}\Big)dy$$

Inside Int. $\longrightarrow$ $u=x^3$

$$du=3x^2\,dx$$

That doesn't help! This is hard! Try: Switching Limits

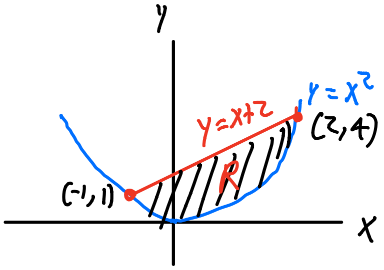
$$\left.\begin{matrix} 0\le y\le 2x \\ 0\le x\le 1\end{matrix}\right\} \int_{x=0}^{1}\Big(\int_{y=0}^{2x} ye^{x^3}\,dy\Big)\,dx = \int_{x=0}^{1} \frac{y^2}{2}e^{x^3}\Big|_{y=0}^{2x}\,dx$$

$$=\frac{1}{2}\int_{x=0}^{1} 4x^2e^{x^3}\,dx = 2\int_{x=0}^{1} x^2e^{x^3}\,dx$$

$$u=x^3 \rightarrow du=3x^2dx \longrightarrow \frac{du}{3}=x^2dx$$

$$\longrightarrow \frac{2}{3}\int e^u\,du = \frac{2}{3}\Big[e^{x^3}\Big]_0^1 = \boxed{\frac{2}{3}(e-1)}$$

Ex Find area of region enclosed by $y=x^2$ and $y=x+2$.
Use $dy\,dx$ then $dx\,dy$.



$x^2=x+2$

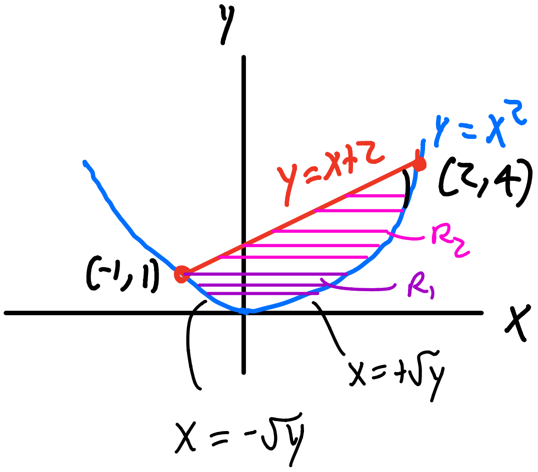
$x^2-x-2=0$

$(x+1)(x-2)=0$

$x=-1,\ x=2$

$$\boxed{A_R = \int_{x=-1}^{2}\int_{y=x^2}^{x+2} dy\,dx}$$

Now integrate w/ rspct to x then y...



We will need 2 Integrals to do this as the lower bound function changes!

$$A_{R_1}+A_{R_2} = \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$\boxed{= \int_{y=0}^{1}\int_{x=-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{y=1}^{4}\int_{x=y-2}^{\sqrt{y}} dx\,dy.}$$

Ex Using a double integral, compute the volume below the surface $z = xy\sin(x^2y)$ for $0 \le x \le \pi$ and $0 \le y \le \pi/2$.

$$\text{Vol} = (\text{area base})(\text{height}) = dx\,dy \cdot z = dx\,dy \cdot \sin(x^2y) \cdot xy$$

$$= \int_0^{\pi/2}\int_0^{\pi} xy\sin(x^2y)\,dx\,dy \Longrightarrow u = x^2y,\ du = 2xy\,dx$$

$$\int_{x=0}^{\pi} xy\sin(x^2y)\,dx = \frac{1}{2}\int \sin(u)\,du = \frac{-1}{2}\cos(x^2y)\Big|_{x=0}^{\pi} = \frac{-1}{2}\left[\cos(\pi^2y) - 1\right]$$

$$\longrightarrow -\frac{1}{2}\int_{y=0}^{\pi/2}\left(\cos(\pi^2y) - 1\right)dy = -\frac{1}{2}\int_0^{\pi/2}\cos(\pi^2y)\,dy + \frac{1}{2}\int_0^{\pi/2}dy = -\frac{1}{2}\sin(\pi^2y)\cdot\frac{1}{\pi^2}\Big|_{y=0}^{\pi/2} + \frac{\pi}{4}$$

$$\boxed{\text{Vol} = -\frac{1}{2\pi^2}\sin\left(\frac{\pi^3}{2}\right) + \frac{\pi}{4}}$$

$$\approx 0.78$$

$\boxed{\text{Ex}}$ A harder, Challenge problem...

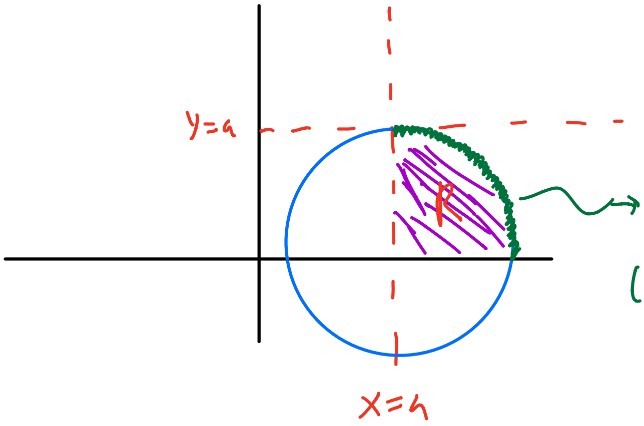
Given the double integral $\int_{y=0}^{a}\int_{x=a}^{a+\sqrt{a^2-y^2}} dx\,dy$: sketch the region and reverse the order of integration.

$a \le x \le a+\sqrt{a^2-y^2} \Longrightarrow$ $x = a+\sqrt{a^2-y^2}$. What does this look like?

$0 \le y \le a$.

It looks like $x = \sqrt{a^2-y^2}$ but shifted +a units rightward

$$x^2 = a^2-y^2 \longrightarrow \underbrace{x^2+y^2=a^2, \text{ and } 0\le y\le a}_{\text{Circle centered at } x=a,\, y=0 \text{ for } 0\le y\le a}$$



$$x = a+\sqrt{a^2-y^2}$$

$$(x-a)^2 = a^2-y^2$$

$$y^2 = a^2-(x-a)^2$$

$$= a^2-(x^2-2ax+a^2)$$

$$y^2 = -x^2+2ax$$

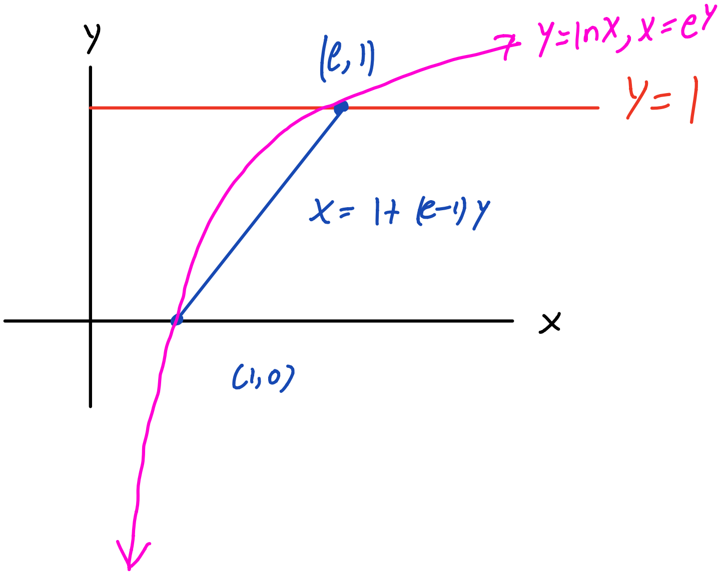
$$y = \underset{-}{\textcircled{+}}\sqrt{2ax-x^2}\ ; \qquad a\le x\le 2a$$

$$\Longrightarrow \boxed{A_R = \int_{x=a}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}} dy\,dx}$$

Ex Reverse the order of integration for the double integral

$$\int_{y=0}^{1}\int_{x=e^{y}}^{1+(e-1)y} g(x,y)\,dx\,dy$$

$$e^{y} \le x \le 1+(e-1)y$$
$$0 \le y \le 1$$



$x = 1+(e-1)y$

$y=0,\ x=1\ ;\ (1,0)$

$y=1,\ x=e\ :\ (e,1)$

$x = e^{y}$

$y = \ln x$

$(\ln(1)=0,\ \ln(e)=1)$

$$\boxed{\int_{x=1}^{e}\int_{y=\frac{x-1}{e-1}}^{\ln x} g(x,y)\,dy\,dx}$$

(Ch 4) Triple Integrals

Start here: Double Integrals vs Triple Integrals

$\int_a^b \int_{g_1(y)}^{g_2(y)} dx\,dy$ gives signed area

$\int_a^b \int_{h_1(z)}^{h_2(z)} \int_{g_1(y,z)}^{g_2(y,z)} dx\,dy\,dz$ gives signed Volume

Ex Compute $\int_0^1 \int_\pi^{2\pi} \int_0^y \cos(xy)\,dz\,dy\,dx$

$$= \int_0^1 \underbrace{\int_\pi^{2\pi} y\cos(xy)\,dy\,dx}_{\text{Hard!}} = \int_\pi^{2\pi} \int_0^1 y\cos(xy)\,dx\,dy$$

(by Fubini)

$$= \int_\pi^{2\pi} y\left(\frac{1}{y}\sin(xy)\right)\Big|_{x=0}^{1} dy = \int_\pi^{2\pi} \sin(y)\,dy = -\cos(y)\Big|_\pi^{2\pi} = -\left[\cos(2\pi) - \cos(\pi)\right]$$

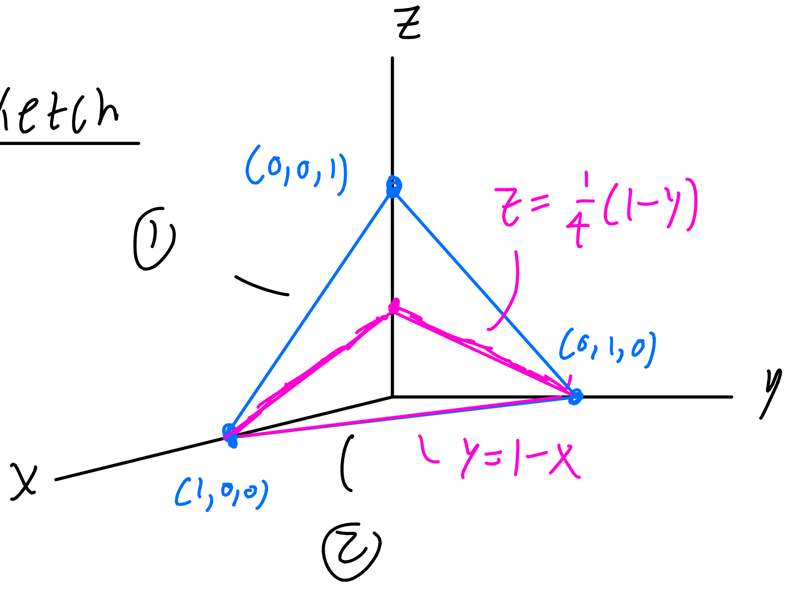
$$= -[2]$$

$$= \boxed{-2}$$

For harder integral problems:
1) Sketch
2) Bounds
3) Integrate

Ex Compute the volume of the region $x \geq 0, y \geq 0, z \geq 0$ which lies between the planes $x+y+z=1$ ①, $x+y+4z=1$ ②

① Sketch



Use Intercepts!

$$P_2 \begin{cases} \underline{x=0}: z=\frac{1}{4}-\frac{1}{4}y \\ \underline{y=0}: z=\frac{1}{4}-\frac{1}{4}x \\ \underline{z=0}: y=1-x \end{cases}$$

② Bounds : What order should we integrate

- Try and use the easiest possible order. Integrating w/ respect to x or y first in this problem would force us to use two integrals. Z will be easiest to start with.

$$\left.\begin{aligned} \tfrac{1}{4}(1-x-y) \leq z \leq 1-x-y \\ 0 \leq y \leq 1-x \\ 0 \leq x \leq 1 \end{aligned}\right\} \int_0^1 \int_0^{1-x} \int_{\frac{1}{4}(1-x-y)}^{1-x-y} dz\, dy\, dx$$

③ Integrate

$$= \int_0^1 \int_0^{1-x} \frac{3}{4}(1-x-y)\, dy dx = \frac{3}{4}\int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Bigg|_{y=0}^{1-x} dx$$

$$= \frac{3}{4}\int_0^1 \left(1-x - x(1-x) - \frac{1}{2}(1-x)^2\right)dx$$

$$= \frac{3}{4}\int_0^1 (1-x-x+x^2 - \frac{1}{2} + x - \frac{1}{2}x^2)dx$$

$$= \frac{3}{4}\int_0^1 (\frac{1}{2} - x + \frac{1}{2}x^2)dx = \frac{3}{4}\left[\frac{1}{2}x - \frac{x^2}{2} + \frac{1}{6}x^3\right]_0^1$$

$$= \frac{3}{4}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]$$

$$= \frac{3}{24} = \boxed{\frac{1}{8}}$$

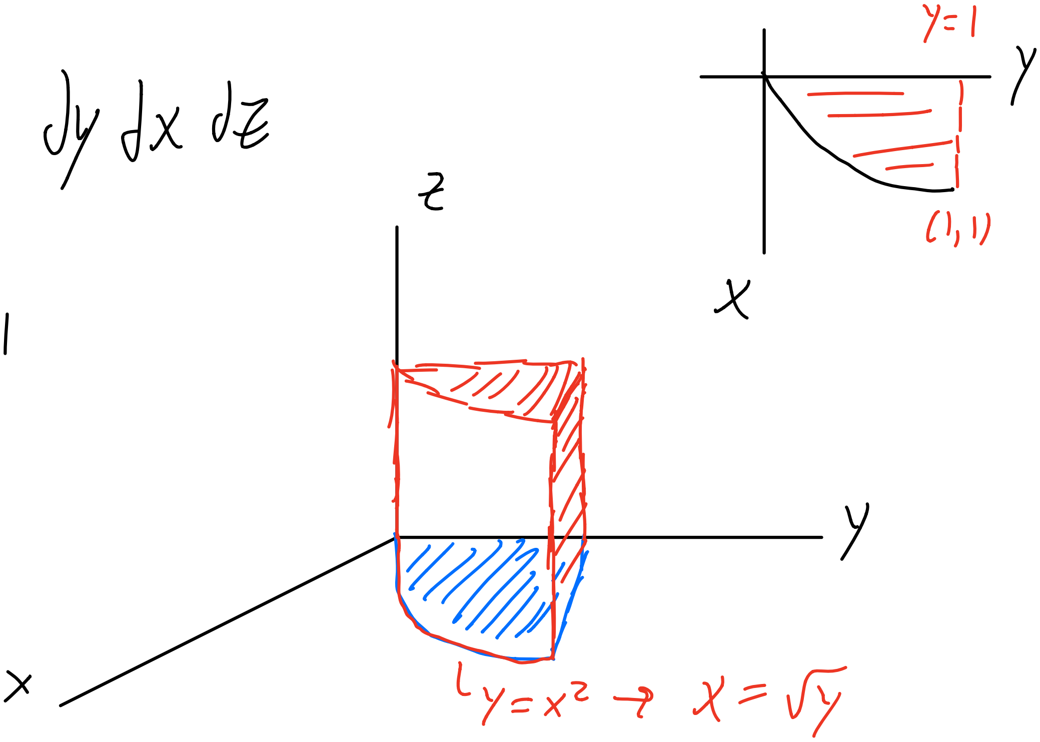
Ex Change the order of integration of the following integral to dx dy dz then compute.

$$\int_0^1\int_0^1\int_{x^2}^1 dy\,dx\,dz$$

$x^2 \le y \le 1$

$0 \le x \le 1$

$0 \le z \le 1$

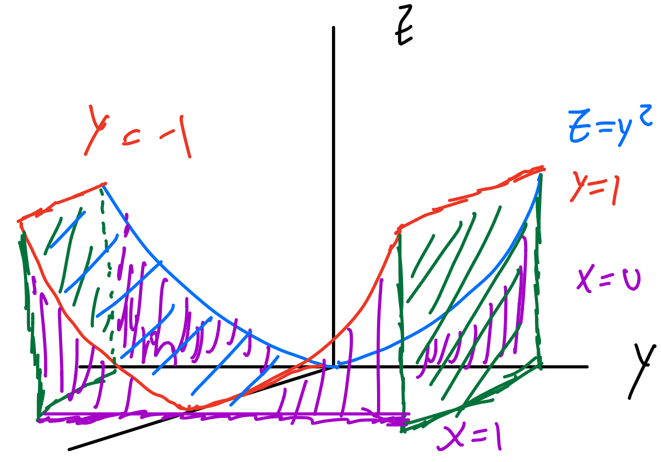


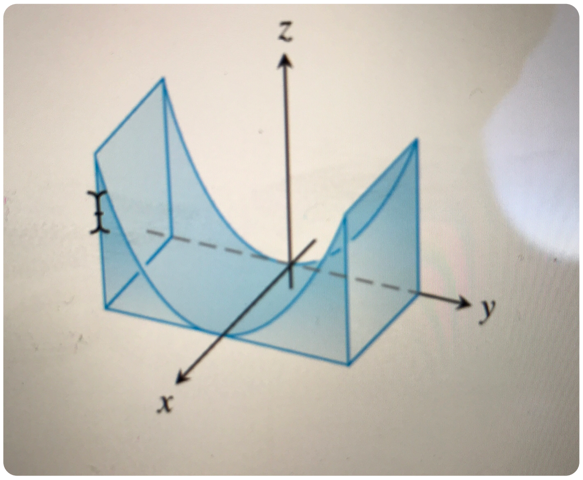
$$\left.\begin{array}{l} 0 \le x \le \sqrt{y} \\ 0 \le y \le 1 \\ 0 \le z \le 1 \end{array}\right\} \text{new order}$$

$$\int_0^1\int_0^1\int_0^{\sqrt{y}} dx\,dy\,dz = \int_0^1\int_0^1 \sqrt{y}\,dy\,dz = \int_0^1 \frac{2}{3}\,y^{3/2}\Big|_0^1 dz$$

$$= \boxed{\frac{2}{3}}$$

Ex Find the volume of the region between the surface $z=y^2$, and the xy plane that is bounded by the planes $x=0$, $x=1$, $y=-1$, $y=1$ (Thomas pg 913. Check ans!)





This solid is symmetric about xy. Let's find volume of RHS and multiply by 2.

$$\text{Vol} = 2 \cdot \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{y^2} dz\, dy\, dx = 2\int_0^1 \int_0^1 y^2\, dy\, dx$$

$$= 2\int_0^1 \frac{y^3}{3}\Big|_0^1 dx$$

$$= \boxed{\frac{2}{3}}$$

(Ch 5, 6) Averages, Centroids

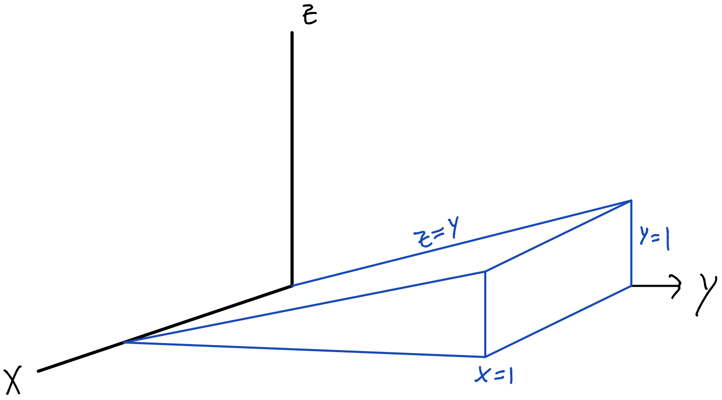
$$\bar{f} = \frac{1}{\int_R d\vec{x}} \int_R f\, d\vec{x}$$

[(area of domain)($\bar{f}$) = Volume(f)] (think sand analogy).

Ex Compute the y coordinate of the COM for the solid object shown: $\rho = y+x$

Note: The object is bounded below by the xy plane

$$\bar{y} = \frac{1}{M}\iiint_R y\rho\, dV$$



$$M = \iiint_R \rho\, dV = \int_0^1\int_0^1\int_0^y (y+x)\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 (xz+yz)\Big|_{z=0}^{y} dy\, dx$$

$$= \int_0^1\int_0^1 (xy+y^2)\, dy\, dx$$

$$= \int_0^1 \frac{x}{2}y^2 + \frac{y^3}{3}\Big|_{y=0}^{1} dx$$

$$= \int_0^1 \left(\frac{x}{2} + \frac{1}{3}\right) dx$$

$$= \frac{x^2}{4} + \frac{1}{3}x \Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

$$\iiint_R y\rho\, dV = \iiint_R y(y+x)\, dz\, dy\, dx = \iiint_R (y^2+xy)\, dz\, dy\, dx$$

$$\int_0^1\int_0^1\int_0^y (y^2+xy)\, dz\, dy\, dx = \int_0^1\int_0^1 (y^2+xy)y\, dy\, dx$$

$$= \int_0^1\int_0^1 (y^3+xy^2)\, dy\, dx = \int_0^1 \frac{y^4}{4} + \frac{xy^3}{3}\Big|_{y=0}^{1} dx$$

$$= \int_0^1 \left(\frac{1}{4} + \frac{x}{3}\right) dx = \frac{1}{4}x\ \frac{1}{6}x^2\Big|_0^1 = \frac{1}{4} + \frac{1}{6}$$

$$= \left(\frac{6}{6}\right)\frac{1}{4} + \left(\frac{4}{4}\right)\frac{1}{6}$$

$$= \frac{6}{24} + \frac{4}{24} = \frac{10}{24} = \frac{5}{12}$$

$$\therefore\ \bar{y} = \frac{1}{7/12}\cdot\frac{5}{12} = \frac{12}{7}\cdot\frac{5}{12} = \boxed{\frac{5}{7}}$$

$$\int_R d\vec{x} = \int_{\text{Cylinder}} dV = V_{\text{Cylinder}} = \underset{\pi r^2 \cdot h}{\pi(2)^2(3)} = \underline{12\pi}$$

$$\int_0^3 \int_{y=-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x(1-y+z)\,dx\,dy\,dz = \int_{z=0}^{3}\int_{y=-2}^{2} \frac{1}{2}(1-y+z)\,x^2 \Big|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy\,dz = \frac{1}{2}\int_{z=0}^{3}\int_{y=-2}^{2}\left(4-y^2-(4-y^2)\right)(1-y+z)$$

$$= \boxed{0}$$

$f(x,y,z) = x(1-y+z)$. Because our cylinder is symmetric w/ rspct to xy, xz, yz planes, the average value of the function over this volume must be 0 because f is directly proportional to X.

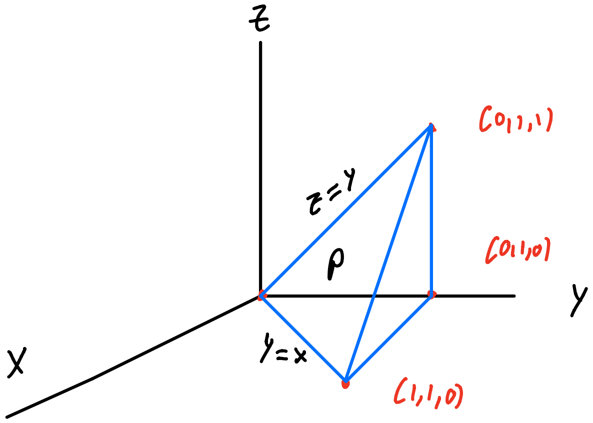
EX The temerature of an irregularly shaped plate at a point (x,y) can be modeled as $T(x,y) = xy^2$. The plate is modeled as the region under the parabola $y=x^2$ for $0 \le x \le 1$. That is $D: \{y: y \le x^2,\ 0 \le x \le 1\}$. Find the avg temperature over this region $\bar{T}$.

$$\bar{T} = \frac{1}{\text{vol}(D)} \int_D T\,d\vec{x} \quad \Big| \quad \text{Vol}(D) = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\int_D T\,d\vec{x} = \int_0^1\int_0^{x^2} xy^2\,dy\,dx = \int_0^1 \frac{xy^3}{3}\Big|_{y=0}^{x^2} dx = \frac{1}{3}\int_0^1 x^7\,dx = \frac{1}{24}$$

$$\bar{T} = \frac{1}{1/3}\cdot\frac{1}{24} = \frac{3}{24} = \boxed{\frac{1}{8}}$$

Ex Suppose the density of an object is $\rho(x,y,z) = xz$. The object occupies the tetrahedron w/ corners $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(0,1,1)$. Setup the integral to find the mass. Tetrahedron is defined uniquely by 4 corners (AKA connect the dots).



$$M = \iiint_R \overset{(\rho\, dV)}{(xz)\, dV}$$

It will be easiest if we can find an eqn of plane P:

$$\vec{n} = (1,1,0) \times (0,1,1) = \begin{vmatrix} \hat{i} & 1 & 0 \\ \hat{j} & 1 & 1 \\ \hat{k} & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \hat{k}$$

$$\vec{n} = 1\hat{i} - 1\hat{j} + 1\hat{k}$$

$$P: \ x - y + z = 0 \rightarrow \underline{z = y - x}$$

$$\boxed{M = \int_0^1 \int_x^1 \int_0^{y-x} xz \, dz\, dy\, dx} \left(= \frac{1}{120}\right).$$

Centroids / Center of mass: $(\bar{x},\bar{y},\bar{z})$

3D solids COM

$$M = \iiint \rho(x,y,z)\,dV$$

$$\bar{x}_i = \frac{1}{M}\iiint_R x_i\,\rho(x,y,z)\,dV$$

If asked for centroid (density unspecified) set $\rho = 1$ and use formulas for $\bar{x},\bar{y},\bar{z}$ (left).

$$\searrow \bar{x} = \frac{m_1x_1+m_2x_2+m_3x_3+\dots}{M}$$

(Discrete case)

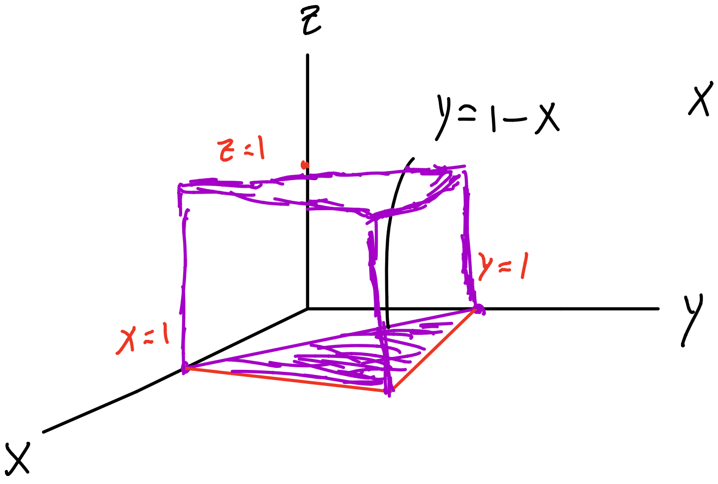
2D plates COM

$$M = \iint \rho(x,y)\,dA$$

$$\bar{x}_i = \frac{1}{M}\iint x_i\,\rho(x,y)\,dA$$

Ex Compute the centroid of the region defined by $0 \le (x,y,z) \le 1$, $x+y \ge 1$.

Lost? Start w/ a sketch



$$x+y \overset{(\ge)}{=} 1$$

$$y \underset{(\ge)}{=} 1-x$$

$$\bar{X} = \frac{1}{M}\iiint x\rho\,dV \quad \underset{\text{Ar } \blacktriangle = \frac{1}{2}(a)(b)}{\underline{w/\ \rho = 1}} \quad (\text{centroid})$$

$$M = \iiint dV = \text{Volume} = \overbrace{\underbrace{\tfrac{1}{2}(1)(1)}_{\text{area of base}}} \cdot \underset{\substack{\uparrow \\ \text{height}}}{1} = \frac{1}{2}$$

$$\int_0^1\int_0^1\int_{1-y}^1 x\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \frac{x^2}{2}\Big|_{1-y}^{1} dy\,dz = \frac{1}{2}\int_0^1\int_0^1 \left(1-(1-y)^2\right)dy\,dz = \frac{1}{2}\int_0^1\int_0^1\left[1-(1-2y+y^2)\right]dy\,dz$$

$$= \frac{1}{2}\int_0^1\int_0^1(2y-y^2)\,dy\,dz = \frac{1}{2}\int_0^1\left(y^2-\frac{y^3}{3}\Big|_0^1\right)dz = \frac{1}{2}\int_0^1\left(\frac{2}{3}\right)dz = \frac{2}{3} \Rightarrow \boxed{\bar{X} = \frac{2}{3}}$$

$\boxed{\bar{y} = \frac{2}{3}}$ By symmetry, $\bar{X} = \bar{y}$

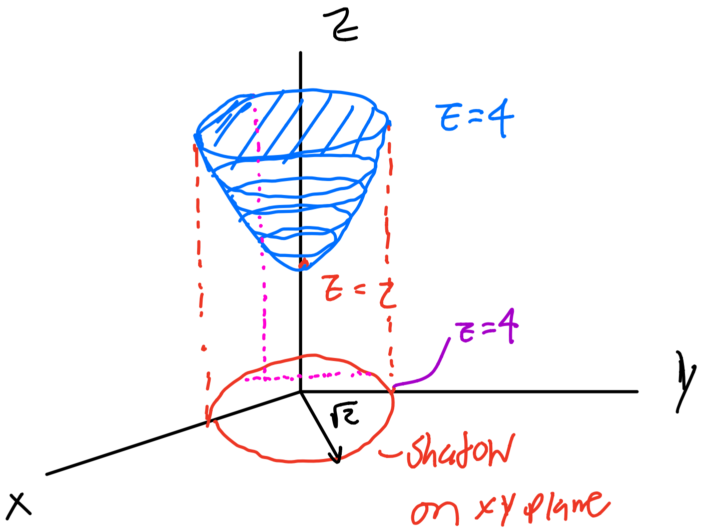
↳ w/ respect to $y = x$

(Looks the same peering in from +x as +y)

$\boxed{\bar{z} = \frac{1}{2}}$ (Think unit cube)

Ex Setup the integrals to solve for the X centroid of the solid object defined by $2 \leq z = x^2+y^2+2 \leq 4$.

Start with Sketch



z

$$\underbrace{\begin{aligned} x^2+y^2+2 &= 2 \rightarrow x^2+y^2 = 0 \\ x^2+y^2+2 &= 3 \rightarrow x^2+y^2 = 1 \\ x^2+y^2+2 &= 3.5 \rightarrow x^2+y^2 = 1.5 \\ x^2+y^2+2 &= 4 \quad x^2+y^2 = 2 \end{aligned}}_{(z-2) = x^2+y^2}$$

Paraboloid!

$$\boxed{\bar{X} = \frac{1}{\int_R d\vec{x}} \int_R x \, d\vec{x} = \frac{1}{\iiint_R dz\,dy\,dx} \iiint_R x \, dz\,dy\,dx}$$

$z = x^2+y^2+2$

$$\boxed{\begin{aligned} &\text{Let } R \text{ be the set: } x^2+y^2+2 \leq z \leq 4 \\ &\qquad -\sqrt{2-x^2} \leq y \leq \sqrt{2-x^2} \\ &\qquad -\sqrt{2} \leq x \leq \sqrt{2} \end{aligned}}$$

Set $z=4$ to determine y bounds

$4 = x^2+y^2+2$

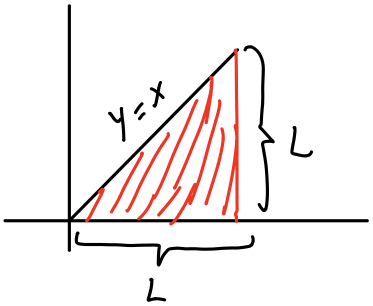
$$\underbrace{x^2+y^2 = 2}_{\text{Circle, radius } \sqrt{2}}$$

(Ch 7) Moments of Inertia

$$I = \int r^2 dm \; ; \; dM = \rho\, dV$$

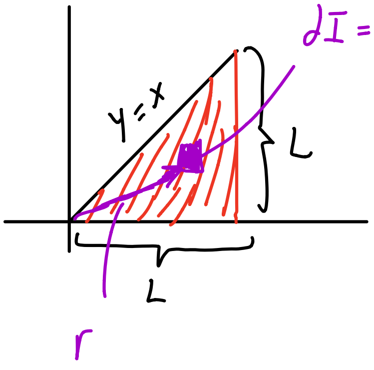
Ex | Consider rotation of the 2 dimensional solid object about the Z-axis. The object is bounded by the lines $y=x$, $y=0$, and $x=L$ where $L>0$. The object has constant density α.

(a) What is the mass M of the object?



$$M = \alpha \cdot A = \alpha \cdot \tfrac{1}{2}(L)(L)$$
$$= \boxed{\frac{\alpha}{2}L^2}$$

(b) What is the moment of inertia element dI?



$$dI = r_z^2 dm = (x^2+y^2)\cdot\alpha\, dx\, dy$$

$$\boxed{dI = \alpha(x^2+y^2)\,dx\,dy}$$

(c) Calculate the moment of inertia about the Z-axis in terms of M.

$$I = \int dI = \alpha \int_0^L \int_0^x (x^2+y^2)\,dy\,dx = \alpha \int_0^L \left(x^2 y + \frac{y^3}{3}\right)\Big|_{y=0}^{x} dx = \alpha \int_0^L \left(x^3 + \frac{x^3}{3}\right) dx$$

$$= \frac{4\alpha}{3}\int_0^L x^3 dx = \frac{\alpha}{3} x^4\Big|_0^L = \frac{\alpha}{3}L^4 \rightarrow M = \frac{\alpha}{2}L^2$$

$$\alpha = \frac{2M}{L^2}$$

$$\Longrightarrow I = \frac{2}{3}\frac{M}{L^2}L^4$$

$$\boxed{I = \frac{2}{3}ML^2}$$

$$I = I_{cm} + Mp^2$$

$$\bar{x} = \frac{1}{\frac{\alpha}{2}L^2}\int_0^L\int_0^x x\alpha\, dy\, dx$$

$$= \frac{\alpha}{M}\int_0^L xy\Big|_0^x dx$$

$$= \frac{\alpha}{m}\int_0^L x^2 dx = \frac{\alpha}{3m} x^3\Big|_0^L$$

$$\bar{x} = \frac{\alpha}{3m}L^3 = \frac{\cancel{\alpha}}{3\frac{\cancel{\alpha}}{2}L^2}L^3 = \frac{2}{3}L$$

$$\bar{y} = \frac{\alpha}{M}\int_0^L\int_0^x y\, dy\, dx$$

$$= \frac{\alpha}{2M}\int_0^L y^2\Big|_0^x dx = \frac{\alpha}{2M}\int_0^L x^2 dx = \frac{\alpha}{6M}L^3$$

$$= \frac{\cancel{\alpha}}{6\frac{\cancel{\alpha}}{2}L^2}L^3$$

$$= \boxed{\frac{L}{3}}$$

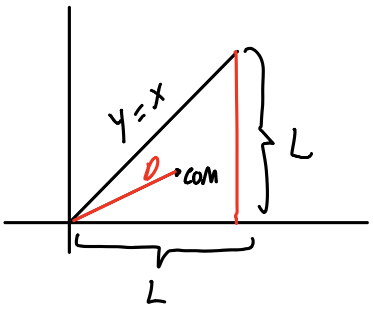
(d) If the COM of the object is $(\bar{x},\bar{y}) = \left(\frac{2}{3}L, \frac{1}{3}L\right)$, compute the inertia about the COM without using an integral.

$$I = I_{cm} + M\underset{\substack{\uparrow \\ \text{dist from} \\ \text{COM}}}{D}^2 \longrightarrow I_{cm} = I - MD^2$$

$$I_{cm} = I_z - MD_z^2 = \frac{2}{3}ML^2 - M\left(\left(\frac{2}{3}L\right)^2 + \left(\frac{1}{3}L\right)^2\right)$$

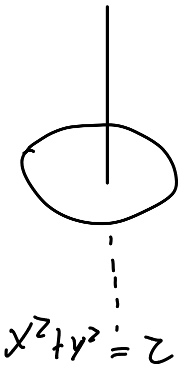
$$= \frac{2}{3}ML^2 - M\left(\frac{4}{4} + \frac{1}{9}\right)L^2$$

$$= \frac{2}{3}ML^2 - \frac{5}{9}ML^2$$



$$\boxed{I_{cm} = \frac{1}{9}ML^2}$$

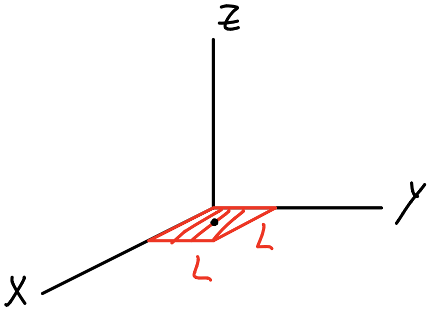
Ex | Set up the integral(s) to compute the moment of inertia of a circular plate of of radius 2, const. density α, about the central axis (⊥ to the plate).

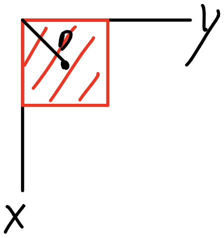


$$dI = r^2 dm = \alpha r^2 dy dx$$

$$\boxed{I = \int dI = \alpha \int_{-2}^{2}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (x^2+y^2)\, dy\, dx}$$

Ex The moment of inertial for a square with uniformly distributed mass M and side length L about a corner is $I_{corner} = \frac{2}{3}ML^2$. What is the moment of inertia about the COM?





$$D = \frac{L\sqrt{2}}{2}$$

$$I_{corner} = I_{cm} + MD^2$$ (parallel axis thrm)

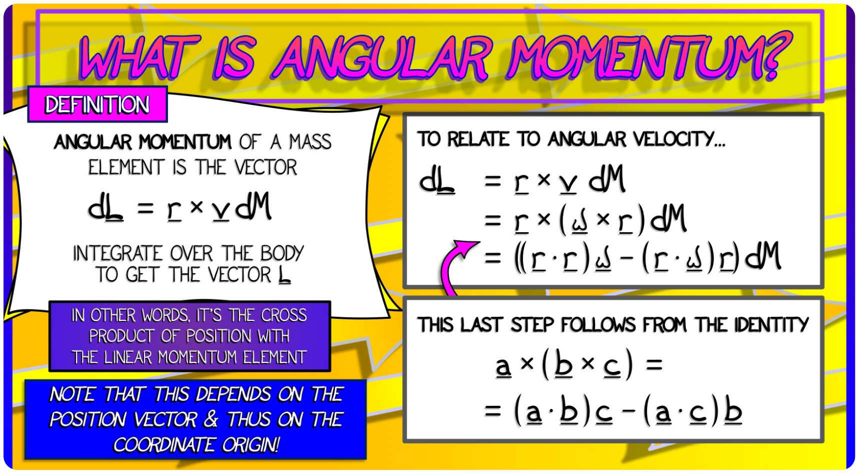
$$I_{cm} = I_{corner} - MD^2$$

$$= \frac{2}{3}ML^2 - M\left(\frac{L^2}{4}\cdot 2\right)$$

$$\boxed{I_{cm} = \frac{1}{6}ML^2}$$

(Ch8) The Inertia matrix

Derivation



WHAT IS ANGULAR MOMENTUM?

DEFINITION

ANGULAR MOMENTUM OF A MASS ELEMENT IS THE VECTOR

$$d\underline{L} = \underline{r} \times \underline{v}\, dM$$

INTEGRATE OVER THE BODY TO GET THE VECTOR $\underline{L}$

IN OTHER WORDS, IT'S THE CROSS PRODUCT OF POSITION WITH THE LINEAR MOMENTUM ELEMENT

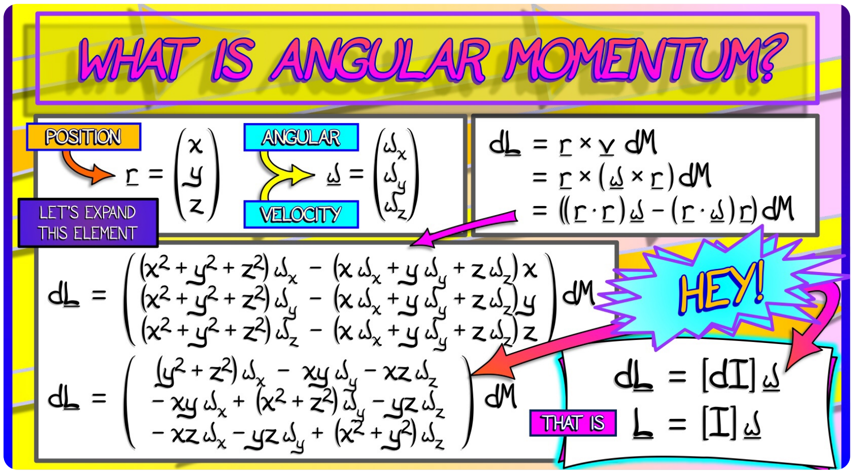
NOTE THAT THIS DEPENDS ON THE POSITION VECTOR & THUS ON THE COORDINATE ORIGIN!

TO RELATE TO ANGULAR VELOCITY...

$$\begin{aligned} d\underline{L} &= \underline{r} \times \underline{v}\, dM \\ &= \underline{r} \times (\underline{\omega} \times \underline{r})\, dM \\ &= ((\underline{r}\cdot\underline{r})\underline{\omega} - (\underline{r}\cdot\underline{\omega})\underline{r})\, dM \end{aligned}$$

THIS LAST STEP FOLLOWS FROM THE IDENTITY

$$\begin{aligned} &\underline{a} \times (\underline{b} \times \underline{c}) = \\ &= (\underline{a}\cdot\underline{b})\underline{c} - (\underline{a}\cdot\underline{c})\underline{b} \end{aligned}$$



WHAT IS ANGULAR MOMENTUM?

POSITION $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

LET'S EXPAND THIS ELEMENT

ANGULAR VELOCITY $\underline{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$

$$\begin{aligned} d\underline{L} &= \underline{r} \times \underline{v}\, dM \\ &= \underline{r} \times (\underline{\omega} \times \underline{r})\, dM \\ &= ((\underline{r}\cdot\underline{r})\underline{\omega} - (\underline{r}\cdot\underline{\omega})\underline{r})\, dM \end{aligned}$$

$$d\underline{L} = \begin{pmatrix} (x^2+y^2+z^2)\omega_x - (x\omega_x + y\omega_y + z\omega_z)x \\ (x^2+y^2+z^2)\omega_y - (x\omega_x + y\omega_y + z\omega_z)y \\ (x^2+y^2+z^2)\omega_z - (x\omega_x + y\omega_y + z\omega_z)z \end{pmatrix} dM$$

$$d\underline{L} = \begin{pmatrix} (y^2+z^2)\omega_x - xy\,\omega_y - xz\,\omega_z \\ -xy\,\omega_x + (x^2+z^2)\omega_y - yz\,\omega_z \\ -xz\,\omega_x - yz\,\omega_y + (x^2+y^2)\omega_z \end{pmatrix} dM$$

HEY!

$$d\underline{L} = [dI]\,\underline{\omega}$$

THAT IS $\underline{L} = [I]\,\underline{\omega}$

$$[I] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$r_x^2 = y^2+z^2$

$r_y^2 = x^2+z^2$

$r_z = x^2+y^2$

↑

Where $I_{xx} = \int r_x^2\, dM$; $I_{yy} = \int r_y^2\, dM$, $I_{zz} = \int r_z^2\, dm$

$$I_{xy} = I_{yx} = -\int xy\, dM$$

$$I_{xz} = I_{zx} = -\int xz\, dm$$

$$I_{yz} = I_{zy} = -\int yz\, dm$$

What's going on here? The inertia matrix is defined to reduce the angular momentum formula L into a nice compact notation.

Recall $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$ ($mvr \sin\theta$)

If you have a massive extended body, you need to add up ang. momentum of all pieces.

$$\longrightarrow \quad d\vec{L} = \vec{r} \times dm\vec{v} = \vec{r} \times (\overbrace{\vec{\omega} \times \vec{r}}^{\vec{v}})\, dm$$

For $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

you eventually get

$$d\vec{L} = \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} dm$$

distribute dm into matrix, integrate each element...

$$d\vec{L} = [dI]\vec{\omega}$$

$$\Longrightarrow \vec{L} = [I]\vec{\omega}$$

Okay... We also have the property that

(*)

$$I_{\vec{u}} = \vec{u} \cdot [I]\vec{u} \qquad ; \ \vec{u} : \text{unit vector}$$

$$I_{\vec{u}} = \vec{u}^T [I] \vec{u} \qquad \longrightarrow \text{weighted avg}$$

This means some objects are "easier" or "harder" to rotate than others.

Why is (✱) true?

$$I_{\vec{u}} = \vec{u}^T [I] \vec{u}$$

$[I]$ is defined from the equation $\vec{L} = [I]\vec{\omega}$ transforming ang. velocity to ang. momentum,

Suppose $\text{dir}(\vec{L}) = \text{dir}(\vec{\omega})$.

Now we can reduce the inertia matrix into a single scalar multiple $L = I\omega$ (not vectorized).

Suppose $\vec{\omega} = \omega\hat{u}$ and $\vec{L} = L\hat{u}$

$$\vec{L} = [I]\vec{\omega}$$

$$L\hat{u} = [I]\omega\hat{u}.$$

Now take dot prod of each side w/ $\hat{u}$

$$\hat{u}\cdot(L\hat{u}) = \hat{u}\cdot([I]\omega\hat{u})$$

$$L\underbrace{\hat{u}\cdot\hat{u}}_{1} = \hat{u}\cdot([I]\omega\hat{u})$$

($\hat{u}$ = unit vector)

$$\Rightarrow L = \underbrace{\hat{u}\cdot([I]\hat{u})}\omega$$

$$\boxed{I_{\hat{u}} = \hat{u}\cdot[I]\hat{u}}$$

(Inertia about some other axis in $\hat{u}$ direction)

$\boxed{\text{Ex}}$

(a) For a square plate shown in the xy plane w/ uniformly distributed mass M and side length L, compute the density ρ.

$$\rho = \frac{M}{A} = \frac{M}{L^2} \longrightarrow \boxed{\rho = \frac{M}{L^2}}$$

(b) Compute the inertia matrix $[I]$

$$[I] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yz} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Start w/ diagonal:

$$dI_{xx} = r_x^2 dm \quad [r_x: \text{dist from } x \text{ axis to mass element}]$$

$$= y^2 dm \;;\; dm = \rho\, dA = \frac{M}{L^2} dx\, dy$$

$$dI_{xx} = y^2 \frac{M}{L^2} dx\, dy$$

$$I_{xx} = \int_R dI_{xx} = \frac{M}{L^2} \int_0^L \int_0^L y^2 dx\, dy = \frac{M}{L^2} \int_0^L y^2 L\, dy = \frac{M}{L} \left.\frac{y^3}{3}\right|_0^L = \boxed{\frac{ML^2}{3}}$$

$$I_{yy} = \frac{M}{L^2} \int_0^L \int_0^L x^2 dy\, dx = I_{xx} \quad (\text{inertia abt } x = \text{inertia abt } y)$$

$$\boxed{I_{xx} = I_{yy} = \frac{1}{3} ML^2}$$

$$dI_{zz} = r_z^2 dm = (x^2+y^2)\, \rho\, dx\, dy$$

$$I_{zz} = \frac{M}{L^2} \int_0^L \int_0^L (x^2+y^2)\, dx\, dy = \boxed{\frac{2}{3} ML^2}$$

Mixed Moments:

$$I_{xy} = -\int_R xy\,dm = \frac{-M}{L^2}\int_0^L\int_0^L xy\,dx\,dy = \frac{-1}{4}mL^2$$

$$\boxed{I_{xy} = I_{yx} = \frac{-1}{4}mL^2}$$

$$I_{xz} = -\int_R xz\,dm; \quad z = 0 \text{ for every } dm$$

For a square $z=0$ $\Big|$ $\boxed{\begin{array}{l} I_{xz} = I_{zx} = 0 \\ I_{yz} = I_{zy} = 0 \end{array}}$

$$\boxed{[I] = \begin{bmatrix} \frac{1}{3}mL^2 & -\frac{1}{4}mL^2 & 0 \\ -\frac{1}{4}mL^2 & \frac{1}{3}mL^2 & 0 \\ 0 & 0 & \frac{2}{3}mL^2 \end{bmatrix}}$$

(C) Suppose $M = 1kg$, $L = 1m$, compute the inertia about the digonal $x=y=z$.

$I_{\vec u}$ is inertia about $\vec u$ axis: $\vec u$ is unit.

$$\vec u = \frac{1}{\sqrt3}(\hat i + \hat j + \hat k) = \frac{1}{\sqrt3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\underline{I_{\vec u} = \vec u^T [I] \vec u = \vec u \cdot [I] \vec u} \quad \text{(weighted average)}$$

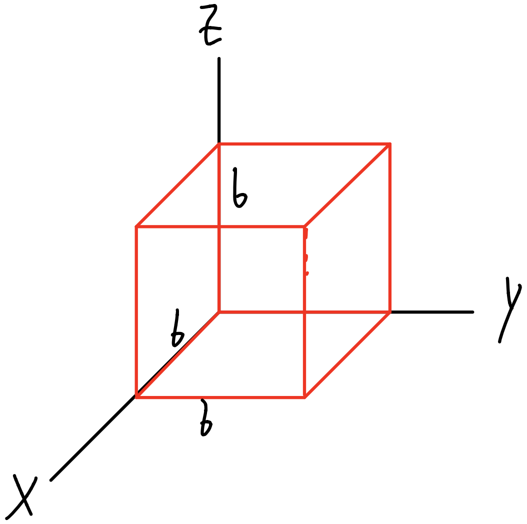
$$I_u = \frac{1}{\sqrt3}\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}\frac{1}{\sqrt3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}\frac{1}{12} & \frac{1}{12} & \frac{2}{3}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\boxed{I_{\vec u} = \frac{5}{18}}$$

(a)

Ex Compute the inertia tensor of a cube (side length b) in the first octant with a corner centered at the origin. Assume uniform mass M.



$$\rho = \frac{m}{v} = \frac{M}{b^3};\quad dm = \rho dV$$

$$I_{xx} = I_{yy} = I_{zz}$$

$$I_{zz} = \rho\int_0^b\int_0^b\int_0^b (x^2+y^2)\,dx\,dy\,dz$$

$$= \rho\frac{2b^5}{3} = \frac{M}{b^3}\frac{2b^5}{3} = \frac{2mb^2}{3}$$

$$\boxed{I_{xx} = I_{yy} = I_{zz} = \frac{2}{3}Mb^2}$$

$$I_{xy} = I_{yx} = -\rho\int_0^b\int_0^b\int_0^b xy\,dx\,dy\,dz = -\rho\frac{b^5}{4} = \frac{M}{b^3}\frac{b^5}{4} = -\frac{M}{4}b^2$$

Note that the cube is symmetrical w/ $x, y, z \geq 0$ extending the same amt in each dir. so

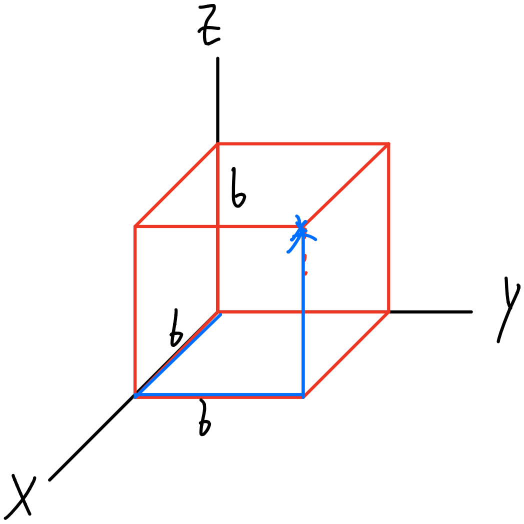
$$\boxed{I_{xy} = I_{xz} = I_{yz} = -\frac{1}{4}Mb^2}$$

$$\boxed{[I] = \begin{bmatrix} \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 \end{bmatrix}}$$

(b) Compute inertia of the above cube with $b=1$, $m=1$ about an axis extending from the origin to the farthest corner.

$$\vec{V} = b\hat{i} + b\hat{j} + b\hat{k}$$

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{b\hat{i}+b\hat{j}+b\hat{k}}{\sqrt{3b^2}} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$



$$I_u = \vec{u} \cdot [I]\vec{u}$$

$$= \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{bmatrix}\frac{2}{3} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1/6\\1/6\\1/6\end{bmatrix} = \frac{1}{18}(3) = \boxed{\frac{1}{6}}$$

(Ch 10-11) Probability (Everything you might need to know)

- ρ is a probability density

Probability per unit something

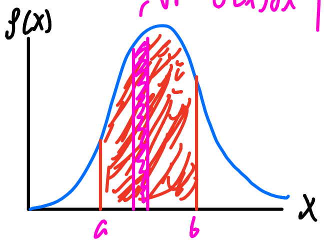
$dP = \rho\, dx$ is a small change in probability

Add up all dPs, you get 1.00

$$\Rightarrow \int_{-\infty}^{\infty} \rho(x)\,dx = 1$$

To compute probability that your "choice" will fall w/in a certain range of values:

$$dP = \rho(x)dx \;\Big|\; P(X \in [a,b]) = \int_a^b p(x)\,dx \;(<1)$$



Higher Dim Prob. density: Formulas and Examples

A random variable $\vec{X}$ in $\mathbb{R}^n$ has probability density $\rho: \mathbb{R}^n \to \mathbb{R}$

$$\boxed{\int_{\text{Entire Region}} \rho(\vec{x})\,d\vec{x} = 1} \;;\; \rho(X \in A) = \int_A dP = \int_A \rho(\vec{x})\,d\vec{x} \quad (\text{MASS})$$

Ex | Find the value of c for which $\rho(x,y) = x + cy^2$ over $R = \{0 \le (x,y) \le 1\}$ is a $\underbrace{\text{joint}}_{\text{dependent on two variables (or more).}}$ PDF.

A joint probability.

For PDF, we must have

$$\iint_R f(x,y)\,dx\,dy = 1$$

$$\int_0^1\int_0^1 (x+cy^2)\,dx\,dy = \int_0^1 \frac{x^2}{2}+cy^2x\Big|_{x=0}^{1} dy = \int_0^1 \left(\frac{1}{2}+cy^2\right)dy$$

$$= \frac{1}{2}y+\frac{c}{3}y^3\Big|_0^1 = \frac{1}{2}+\frac{c}{3} = 1$$

$$\frac{c}{3} = \frac{1}{2}$$

$$\boxed{c = 3/2}$$

What is the Probability that $X<Y$ (Biggest X can be is Y)

$$P(X<Y) = \int_0^1\int_0^y \left(x+\frac{3}{2}y^2\right)dx\,dy \quad \left(= \int_0^1\int_x^1 \left(x+\frac{3}{2}y^2\right)dy\,dx \right)$$

$$= \int_0^1 \frac{x^2}{2}+\frac{3}{2}y^2x\Big]_{x=0}^{y} dy$$

$$= \int_0^1 \left(\frac{y^2}{2}+\frac{3}{2}y^3\right)dy$$

$$= \frac{y^3}{6}+\frac{3}{8}y^4\Big|_0^1$$

$$\boxed{P(x<y) = \frac{1}{6}+\frac{3}{8} = \frac{13}{24}}$$

Expectation: Average Value of random Variables (weighted Avg)

$\longrightarrow$ Center of mass

For $\rho(x,y)$ defined over R...

$$\boxed{\begin{aligned} E(x) &= \iint_R x\,\rho(x,y)\,dx\,dy \quad \left(\Big/ \iint_R \rho(x,y)\,dx\,dy = 1 \right) \\ E(y) &= \iint_R y\,\rho(x,y)\,dx\,dy \end{aligned}}$$

Compute $E(\vec{x})$ for $\rho(x,y) = x^2 + \frac{3}{2}y^2$ on $R = \{0 \leq (x,y) \leq 1\}$

$$E(\vec{x}) = \begin{bmatrix} E(x) \\ E(y) \end{bmatrix} ; \quad E(x) = \int_0^1\!\!\int_0^1 x(x^2 + \tfrac{3}{2}y^2)\,dx\,dy = \boxed{\frac{1}{2}}$$

$$E(y) = \int_0^1\!\!\int_0^1 y(x^2 + \tfrac{3}{2}y^2)\,dx\,dy = \boxed{\frac{13}{24}}$$

Variance: Information about Spread of data

$\longrightarrow$ Inertia: Information about Spread of mass.

High Inertia: mass distributed further from axis, hard to rotate

High Variance: High probability of Selecting data pts far from expected value.

$$\Rightarrow \rho(x,y)\ldots \quad \boxed{V(x,y) = \iint_R \left((x - E(x))^2 + (y - E(y))^2 \right) \rho(x,y)\,dx\,dy .}$$

And $\sigma = \sqrt{V}$ (standard dev).

Random variables X,Y are independent if $\underline{f(x,y) = f_X(X) f_Y(Y)}$

→ Pr[X] is <u>independent</u> of Pr[Y]

For joint PDF $f(x,y)$, if there is a $f_x, f_y \ni f(x,y) = f_x(x) f_y(y)$, X, Y are independent!

Ex $f(x,y) = 12e^{-2x-6y}$ $\xrightarrow{\text{over}}$ $R = \{0 \leq (x,y) \leq \infty\}$. Are x, y independent?

$f(x,y) = 12e^{-2x} e^{-6y} = f_X(x) f_Y(y)$. <u>Yes!</u>

Check w/ $Cov(X,Y) = \int_0^\infty \int_0^\infty (X - E(x))(Y - E(y)) f(x,y)\,dx\,dy = 0$ ✓ $\xrightarrow{X,Y \text{ indep.}}$

Ex $f(x,y) = \frac{1}{16}(x^2 + 2y^2)$ over $R = \{0 \leq (x,y) \leq 2\}$. Are x, y indep?

Is there a $f_x, f_y \ni f(x,y) = f_x f_y$?? No!

Could try

$$\frac{1}{16} x^2 \cdot C = \frac{1}{16} x^2 + \frac{1}{8} y^2$$

$$x^2 C = x^2 + 2y^2$$

$C = 1 + 2\,y^2/x^2 = f_y(y)$?? No!

$C = C(x,y)$. No good.

Here, $cov(x,y) \neq 0$.

Ex The PDF of a neutrino being detected in a flat square bin of side length ℓ is directly proportional to

$f(x,y) \propto x^2 y$; $R = \{0 \leq (x,y) \leq 1\}$ where x,y is a location in the bin.

A) Find the exact relation for f.

$$C\int_0^\ell\int_0^\ell x^2 y\,dx\,dy = C\int_0^\ell \frac{\ell^3}{3} y\,dy = \frac{C\ell^3}{3}\,\frac{\ell^2}{2} = \frac{C\ell^5}{6} = 1$$

$$\boxed{C = \frac{6}{\ell^5}}$$

What is the probability that a neutrino will be detected in the upper half of the box?

$$P(x>y) = \frac{6}{\ell^5}\int_0^1\int_y^1 x^2 y\,dx\,dy = \boxed{\frac{3}{5\ell^2}}$$

Extra: For $\vec{y} = A\vec{x}$

prove $E(\vec{y}) = AE(\vec{x})$

$$E(\vec{y}) = E(A\vec{x}) = A\iint_R \vec{x}\,dR = AE(\vec{x})$$

Prove $[V(\vec{y})] = A[V(\vec{x})]A^T$ (Extra: $\vec{y} = A\vec{x}$)

$$[V(\vec{y})] = E[(\vec{y} - E(\vec{y}))(\vec{y} - E(\vec{y}))^T]$$

$$= E[(A\vec{x} - E(A\vec{x}))(A\vec{x} - E(A\vec{x}))^T]$$

$$= E[A(\vec{x} - E(\vec{x}))(\vec{x} - E(\vec{x}))^T A^T]$$

$$= A\,E[(\vec{x} - E(\vec{x}))(\vec{x} - E(\vec{x}))^T]A^T$$

$$\boxed{[V(\vec{y})] = A\,V(\vec{x})A^T}$$

Ch 12 Not Covered. Inertia matrix = covariance matrix. Basically.

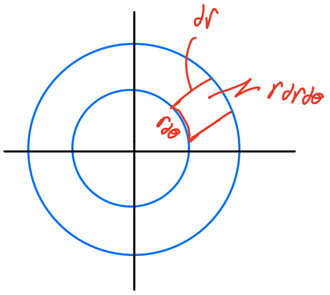
Ch 13 Cylindrical Coordinates

We know polar coordinates...

$x = r\cos\theta$ $\quad r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$dA = r\,dr\,d\theta$



Cylindrical: Same as polar, but mapped up to z axis.

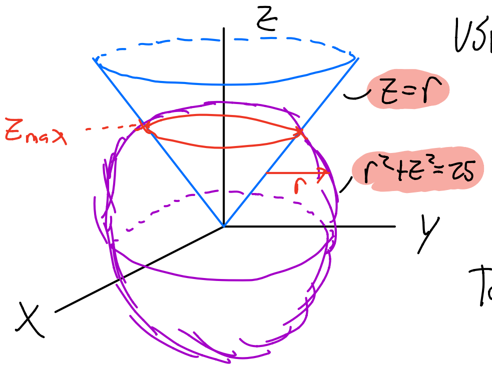
$dV = r\,dr\,d\theta\,dz$

Ex Compute the volume of the region outside the cone, inside the sphere

Cone: $z = (x^2+y^2)^{1/2} \geq 0$

Sphere: $x^2+y^2+z^2 = 25$

Always start w/ a sketch...



Using cylindrical coordinates...

Cone: $z = \sqrt{x^2+y^2} = r$

Sphere: $x^2+y^2+z^2 = r^2+z^2 = 25$

To integrate in spherical coords: we want everything in terms of (r, θ, z).

$z \leq r \leq \sqrt{25-z^2}$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq z_{max}$ ← when cone intersects sphere!

Find z_{max}

Cone: $z = r$; Sphere: $r^2+z^2 = 25$

↳ $r^2 + r^2 = 25$

$r^2 = \frac{25}{2} \rightarrow r = \frac{5}{\sqrt{2}}$ is radius of intersection! ; $z_{max} = \frac{5}{\sqrt{2}}$

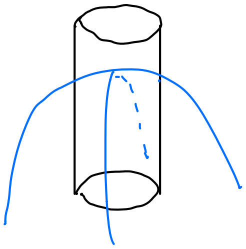
$$\Longrightarrow \int_0^{5/\sqrt{2}}\int_0^{2\pi}\int_z^{\sqrt{25-z^2}} r\,dr\,d\theta\,dz = \frac{1}{2}\int_0^{5/\sqrt{2}}\int_0^{2\pi}(25-z^2-z^2)\,d\theta\,dz = \frac{1}{2}\int_0^{5/\sqrt{2}}(25-2z^2)2\pi\,dz$$

$$\frac{1}{2}-\frac{3}{2} =$$

$$= \pi\left[25z-\frac{2}{3}z^3\right]_0^{5/\sqrt{2}} = \boxed{\pi\left[125/\sqrt{2}-\frac{2}{3}\cdot 125/2^{3/2}\right]} = \boxed{125\frac{\pi}{3}\sqrt{2}}$$

$$\pi\left(\frac{125\sqrt{2}}{2}-\frac{125}{3}\frac{1}{\sqrt{2}}\right) = \pi\left(\frac{125\sqrt{2}}{2}-\frac{125}{3}\frac{\sqrt{2}}{2}\right)$$

Ex Find the volume of the region inside the cylinder $x^2+y^2=1$ bounded above by the paraboloid $z=9-x^2-y^2$; $z\geq 0$.



Cylinder $x^2+y^2=1$

$r^2=1$; $r=1$

Paraboloid : $z=9-x^2-y^2=9-r^2$

$z=9-r^2$

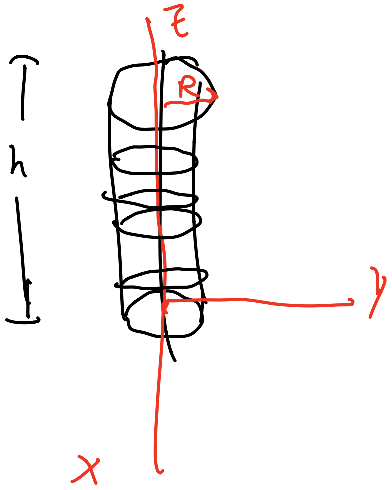
$0\leq z\leq 9-r^2$

$0\leq r\leq 1$

$0\leq\theta\leq 2\pi$.

$$\int^{2\pi}\int^{1}\int^{9-r^2} dz\,r\,dr\,d\theta = \boxed{\frac{17\pi}{2}}$$

Ex Find moment of inertia of cylinder height h, radius R, mass M about central axis. ; $\rho=\frac{M}{V}=\frac{M}{\pi R^2 h}$



$$dI_{zz} = R_z^2\,dm = r^2\,dm = r^2(\rho\,dV)$$

$$= r^2\rho(r\,dr\,d\theta\,dz)$$

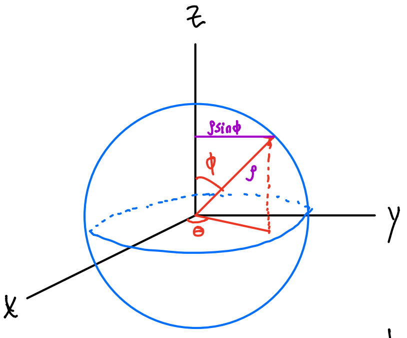
$$= r^3\rho\,dr\,d\theta\,dz \text{ (cylindrical)}.$$

$$I_{zz} = \rho\int_0^h\int_0^{2\pi}\int_0^R r^3\,dr\,d\theta\,dz$$

$$= \frac{M}{\pi R^2 h}\int_0^h\int_0^{2\pi}\frac{R^4}{4}\,d\theta\,dz$$

$$= \frac{MR^2}{4\cancel{\pi}h}2\cancel{\pi}h = \boxed{\frac{1}{2}MR^2}$$

(Ch 14) Spherical coordinates

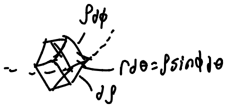


radius in xy plane

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$r = \rho\sin\phi$

$\boxed{x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\cos\theta,\ z = \rho\cos\phi}$; $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$.



$dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$

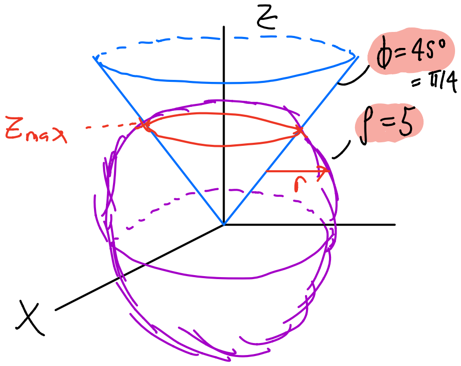
Ex Compute the volume of the region outside the cone, inside the sphere

Cone: $z = (x^2+y^2)^{1/2}$ ≥ 0

Sphere: $x^2+y^2+z^2 = 25$

But now use spherical coords.

Always start w/ a sketch...



$x^2+y^2+z^2 = 25 \to \rho = 5$

$z = \sqrt{x^2+y^2}$

$\rho\cos\phi = \rho\sin\phi$

$\cos\phi = \sin\phi$

$\phi = 45°$

$$V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^5 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

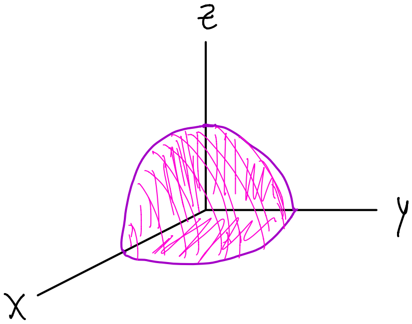
$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{125}{3}\sin\phi\, d\phi\, d\theta$$

$$= \frac{125}{3}\left[\int_0^{2\pi}\left[-\cos\phi\right]_{\pi/4}^{\pi/2} d\theta\right]$$

$$= \frac{2\pi}{3}\cdot 125\left[-\cos(\pi/2) + \cos(\pi/4)\right]$$

$$= \frac{\cancel{2}\pi}{3}\, 125\left[\frac{\sqrt{2}}{\cancel{2}}\right] = \boxed{125\frac{\pi}{3}\cdot\sqrt{2}}$$

Ex Use spherical coordinates to calculate the volume of $x^2+y^2+z^2 = 9$ in the first octant $(x,y,z \geq 0)$.



$$\left.\begin{array}{l} 0 \leq \rho \leq 3 \\ 0 \leq \phi \leq \pi/2 \\ 0 \leq \theta \leq \pi/2 \end{array}\right\} \quad V = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \boxed{\frac{9}{2}\pi}$$

$\boxed{\text{Ex}}$ Suppose the temperature at (x,y,z) is $T(x,y,z) = 1/(x^2+y^2+z^2)$.

Find the average temperature inside the Unit Sphere Centered at the origin.

(a)

$$\overline{T} = \frac{1}{vol(R)}\int_R T\,dv \longrightarrow \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \frac{1}{\rho^2}\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sin\phi\, d\rho\, d\phi\, d\theta$$

$$vol(R) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$$

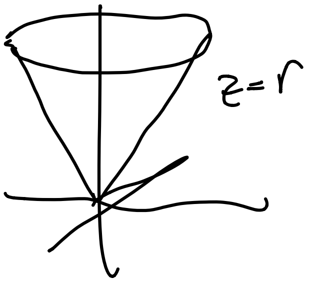
$$= \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 2\pi\left[-\cos(\pi) + \cos(0)\right]$$

$$= 2\pi\left[1+1\right]$$

$$= \boxed{4\pi}$$

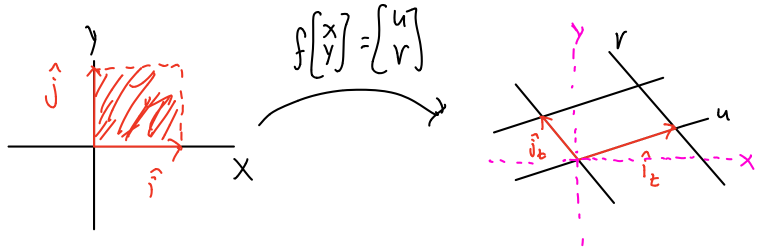
For $z=\sqrt{x^2+y^2}$ $\quad 0 \le z \le 4$

$$\frac{1}{vol(R)}\int_R T\,dv = \frac{1}{\int_0^4\int_0^{2\pi}\int_0^{z} r\,dr\,d\theta\,dz} \int_0^4\int_0^{2\pi}\int_0^{z} \frac{1}{r^2+z^2}\, r\,dr\,d\theta\,dz$$



(Ch 15) Change of Variables

In going from coordinate system $(x_1,x_2) \to (u_1,u_2)$, the area element can change.



$|\det[Df]|$ gives us factor by which area scales going from $(x,y) \to (u,v)$.

$\boxed{\text{Ex}}$ If $\det(A)=3$ and an area is measured in the xy plane as 2, What is the area after applying the linear transformation A?

$\Longrightarrow 2 \cdot 3 = 6$

So, if you can accept this fact...

$$|\det[Df]|\, dx\,dy = du\,dv$$

$$\boxed{|\det[Df]|\, d\vec{x} = d\vec{u} \quad \text{for } f(\vec{x}) = \vec{u}}$$

$$\int_R h(f(\vec{x}))|\det[Df]|\, d\vec{x} = \int_{f(R)} h(\vec{u})\, d\vec{u} \quad ; \; f\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}u\\v\end{bmatrix}$$

Ex Integrate $x^2 - xy + y^2$ over the region $x^2 - xy + y^2 \le 2$ Hard!

Using the transformation $x = \sqrt{2}u - \sqrt{2/3}\,v$, $y = \sqrt{2}u + \sqrt{2/3}\,v$.

$$\longrightarrow x^2 - xy + y^2 = \underline{2u^2} - \cancel{2\sqrt{\tfrac{4}{3}}uv} + \underline{\tfrac{2}{8}v^2} - (\sqrt{2}u - \sqrt{2/3}\,v)(\sqrt{2}u + \sqrt{2/3}\,v) + \underline{2u^2} + \cancel{2\sqrt{\tfrac{4}{3}}uv} + \underline{\tfrac{2}{8}v^2}$$

$$= 4u^2 + \tfrac{4}{8}v^3 - \left(2u^2 - \tfrac{2}{3}v^2\right)$$

$$= \underline{4u^2} + \tfrac{4}{8}v^2 - \underline{2u^2} + \tfrac{2}{8}v^2$$

$$= \underline{2u^2 + 2v^2}.$$

The region of integration: $x^2 - xy + y^2 \le 2$

$u^2 + v^2 \le 1$ (using above result)

We have $x(u,v), y(u,v)$ $\therefore (x,y)\ f\begin{bmatrix} u \\ v \end{bmatrix} \ldots$

Now we almost have everything... $??\,dx\,dy = ??\,du\,dv$

define $\begin{bmatrix} x \\ y \end{bmatrix} = f\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sqrt{2}u - \sqrt{2/3}\,v \\ \sqrt{2}u + \sqrt{2/3}\,v \end{bmatrix}$; $f: (u,v) \to (x,y)$

$\therefore$ $|\det[Df]|$ gives us scale factor from $(u,v) \to (x,y)$.

$$\det[Df]\,du\,dv = dx\,dy$$

$$\det[Df] = \det\begin{bmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{bmatrix} = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}.$$

$$\therefore \iint_{x^2 - xy + y^2 \le 2} (x^2 - xy + y^2)\,dx\,dy = \iint_{u^2 + v^2 \le 1} 2(u^2 + v^2)\,\frac{4}{\sqrt{3}}\,du\,dv$$

$$= \frac{4}{\sqrt{3}} \cdot 2 \int_0^{2\pi}\int_0^1 r^2\, r\,dr\,d\theta$$

$$= \frac{8}{\sqrt{3}} \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1\,d\theta = \frac{8}{4\sqrt{3}} \cdot 2\pi = \boxed{\frac{4\pi}{\sqrt{3}}}$$

Ex Evaluate $\int_0^2 \int_{(x-2)/2}^{(2-x)/2} (3x+2y)\, dy\, dx$

using $u = x+2y$, $v = x-2y$.

First handle integral ...

$\frac{x}{2} - 1 \le y \le 1 - \frac{x}{2}$

$0 \le x \le 2$

Now, introduce u and v...

$u = x+2y \qquad v = x-2y$

solve for $x(u,v)$, $y(u,v)$

$$\begin{array}{r} u = x+2y \\ \underline{v = x-2y} \\ u+v = 2x \end{array}$$

$u = 2x - v \rightarrow \underline{x = \frac{u+v}{2}}$

$v = \frac{u+v}{2} - 2y$

$2y = \frac{u}{2} + \frac{v}{2} - v = \frac{u}{2} - \frac{v}{2}$

$\underline{y = \frac{u-v}{4}}$

$0 \le x \le 2$

$0 \le \frac{u+v}{2} \le 2$

$0 \le u+v \le 4$

$\underline{-u \le v \le 4-u}$

$\frac{u+v}{4} - 1 \le \frac{u-v}{4} \le 1 - \frac{u+v}{4}$

$u+v-4 \le u-v \le 4-u-v$

$v-4 \le -v \le 4-2u-v$

$-v+4 \ge v \ge v+2u-4$

$v \le -v+4$

$2v \le 4$

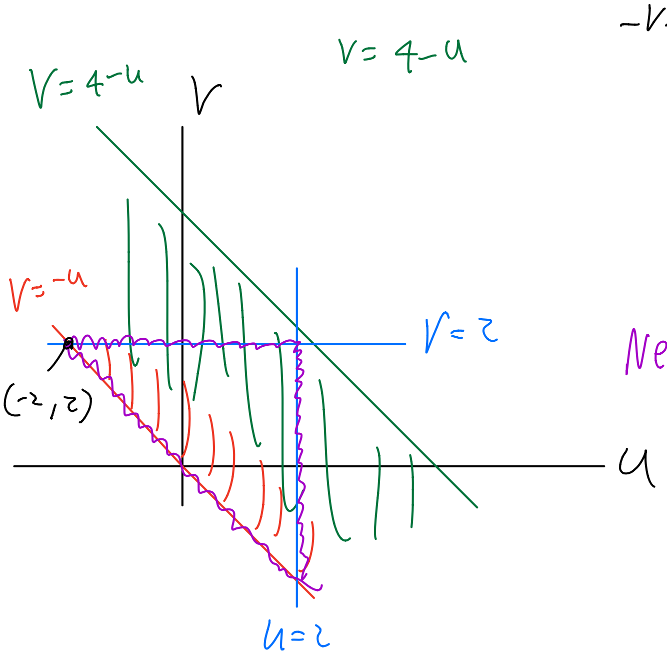
$\underline{v \le 2}$

$v \ge v+2u-4$

$0 \ge 2u-4$

$2u \le 4$

$\underline{u \le 2}$



New region is a ▲. Easy!

Integrand: $3x+2y$

$= 3\left(\frac{u+v}{2}\right) + 2\left(\frac{u-v}{4}\right)$

$= \frac{3}{2}u + \frac{3}{2}v + \frac{u}{2} - \frac{v}{2}$

$= 2u + v$

?? $du\,dv$ = ?? $dx\,dy$

We were given $u(x,y)$, $v(x,y)$: $\begin{bmatrix} u \\ v \end{bmatrix} = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ x-2y \end{bmatrix}$ $\mid f : (x,y) \to (u,v)$

$[Df]$ gives scaling from $(x,y) \to (u,v)$. Goes on xy side!

$$[Df] = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \qquad -2-2$$

$$|\det[Df]| = 4 \qquad 4\,dx\,dy = du\,dv$$

$$dx\,dy = \frac{1}{4}du\,dv$$

$$\frac{1}{4}\int_{-2}^{2}\int_{-u}^{2}(2u+v)\,dv\,du = \boxed{4}$$

Ex $\int_{2}^{6}\int_{1/y}^{3/y} xy^3\,dx\,dy$ Using substitution $x = \frac{v}{6u}$, $y = 2u$.

$$\frac{1}{y} \leq x \leq \frac{3}{y}$$

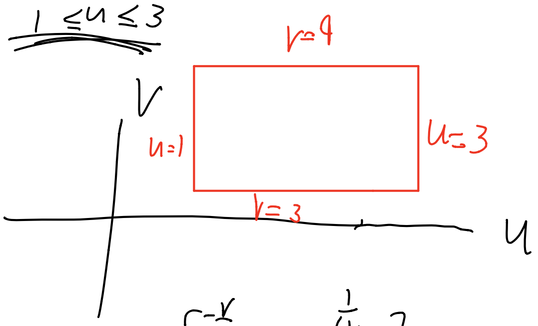
$$2 \leq y \leq 6 \quad \to \quad 2 \leq 2u \leq 6$$

$$\underline{1 \leq u \leq 3}$$

$$\frac{1}{2u} \leq \frac{v}{6u} \leq \frac{3}{2u}$$

$$\frac{1}{2} \leq \frac{v}{6} \leq \frac{3}{2}$$

$$\underline{3 \leq v \leq 9}$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = f\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} v/6u \\ 2u \end{bmatrix} \qquad [Df] = \begin{bmatrix} \frac{-v}{6u^2} & \frac{1}{6u} \\ 2 & 0 \end{bmatrix}$$

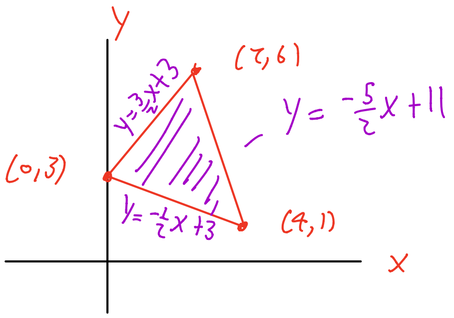
$$\det[Df] = -1/3u. \longrightarrow dx\,dy = +\frac{1}{3u}du\,dv$$

$$\therefore \iint_R xy^3\,dx\,dy = \int_3^9\int_1^3 \frac{v}{6u}\,8u^3 \cdot \frac{1}{3u}\,du\,dv = \frac{4}{9}\int_3^9\int_1^3 uv\,du\,dv = \frac{4}{9}\int_3^9 v\left(\frac{9}{2}-\frac{1}{2}\right)dv = \frac{16}{9}\left(\frac{81}{2}-\frac{9}{2}\right)$$

$$= \frac{16}{9}\left(\frac{72}{2}\right) = \frac{16 \cdot 36}{9}$$

$$= \boxed{64}$$

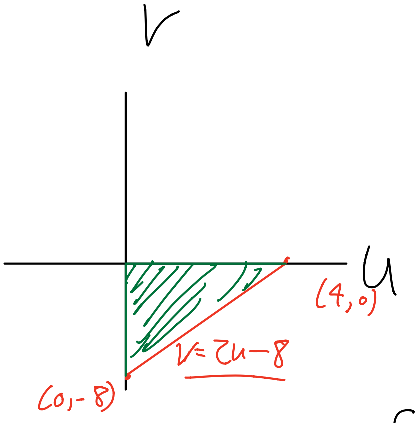
EX | Evaluate $\iint_R 2x\,dA$ where R is the triangle w/ vertices (0,3),(4,1),(2,6) using the Change of vars. $x=\frac{1}{2}(u-v)$, $y=\frac{1}{4}(3u+v+12)$.



$$y=\frac{-5}{2}x+1$$
$$\frac{1}{4}(3u+v+12)=\frac{-5}{2}\left(\frac{1}{2}(u-v)\right)+11$$
$$\frac{3}{4}u+\frac{1}{4}v+3=\frac{-5}{4}u+\frac{5}{4}v+11$$
$$2u=v+8$$
$$\underline{v=2u-8}$$

$$y=\frac{3}{2}x+3$$
$$\frac{1}{4}(3u+v+12)=\frac{3}{2}\left(\frac{u-v}{2}\right)+3$$
$$\cancel{\frac{3}{4}u}+\frac{1}{4}v+\cancel{3}=\cancel{\frac{3u}{4}}-\frac{3v}{4}+\cancel{3}$$
$$\underline{v=0}$$

$$y=-\frac{1}{2}x+3$$
$$\frac{1}{4}(3u+v+12)=\frac{-1}{2}\left(\frac{u-v}{2}\right)+3$$
$$\frac{3}{4}u+\cancel{\frac{1}{4}v}+\cancel{3}=-\frac{u}{4}+\cancel{\frac{v}{4}}+\cancel{3}$$
$$\underline{u=0}$$



$$2u-8\le v\le 0$$
$$0\le u\le 4$$

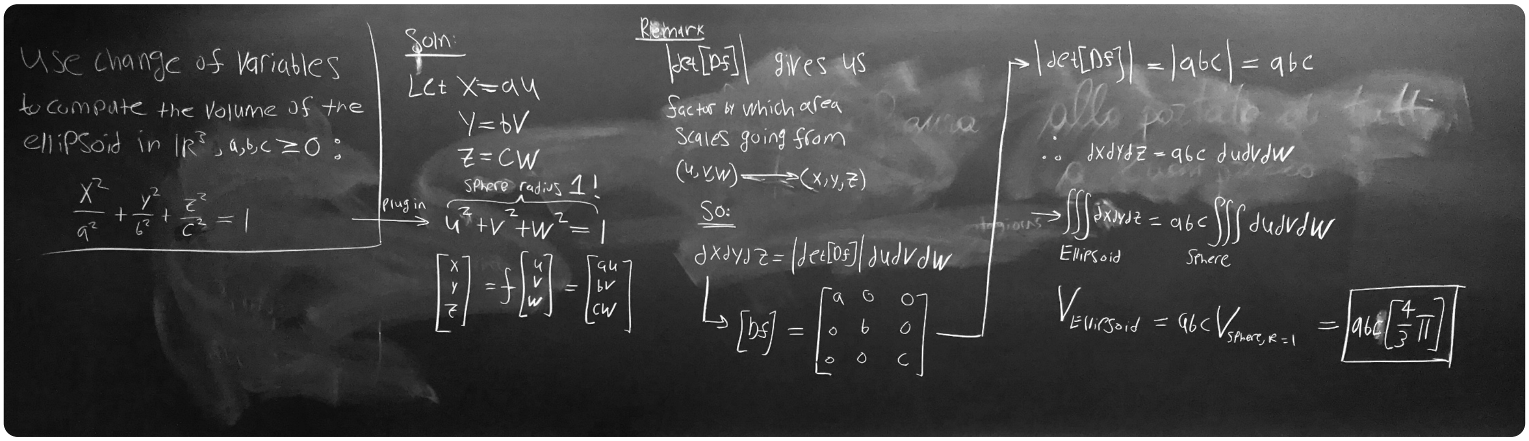
$??\,dx\,dy=??\,du\,dv$

$dx\,dy=\frac{1}{2}du\,dv$

$$\begin{bmatrix}x\\y\end{bmatrix}=f\begin{pmatrix}u\\v\end{pmatrix}=\begin{bmatrix}\frac{1}{2}u-\frac{1}{2}v\\ \frac{3}{4}u+\frac{1}{4}v+3\end{bmatrix};\ [Df]=\begin{bmatrix}1/2&-1/2\\3/4&1/4\end{bmatrix}\xrightarrow{\det}\frac{1}{2}$$

$$\int_0^4\int_{2u-8}^0(u-v)\,\frac{1}{2}\,dv\,du=\boxed{32}$$

Ex



Use change of variables to compute the volume of the ellipsoid in $\mathbb{R}^3$, $a,b,c \geq 0$:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Soln:

Let $x = au$
$y = bv$
$z = cw$

$$\xrightarrow{\text{plug in}} \overbrace{u^2+v^2+w^2=1}^{\text{sphere radius } 1!}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = f\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} au \\ bv \\ cw \end{bmatrix}$$

Remark

$|\det[Df]|$ gives us factor by which area scales going from $(u,v,w) \longrightarrow (x,y,z)$

So:

$$dx\,dy\,dz = |\det[Df]|\,du\,dv\,dw$$

$$\hookrightarrow [Df] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$\rightarrow |\det[Df]| = |abc| = abc$$

$$\therefore\ dx\,dy\,dz = abc\,du\,dv\,dw$$

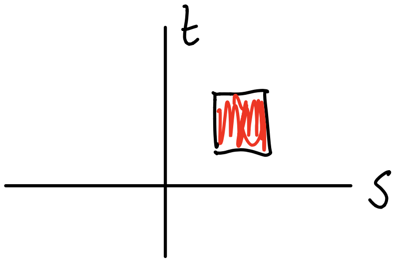
$$\rightarrow \underbrace{\iiint}_{\text{Ellipsoid}} dx\,dy\,dz = abc \underbrace{\iiint}_{\text{Sphere}} du\,dv\,dw$$

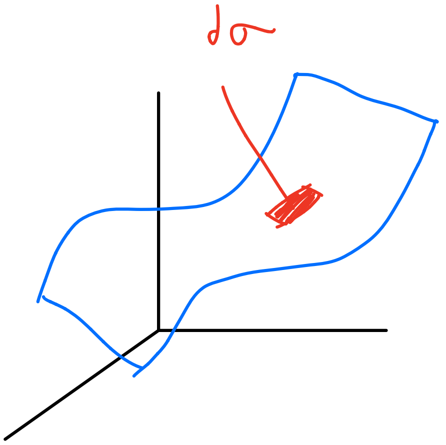
$$V_{\text{Ellipsoid}} = abc\,V_{\text{Sphere}, R=1} = \boxed{abc\left[\frac{4}{3}\pi\right]}$$

(Ch 17) Surface Integrals

Consider a surface parametrized by

$$G\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}x(s,t)\\y(s,t)\\z(s,t)\end{bmatrix}$$





$$d\sigma = \overset{\overset{\text{some factor}}{\uparrow}}{??}\; ds\, dt$$

$$\text{Consider} \quad [DG] = \begin{bmatrix} \frac{\partial x}{\partial s} & \vdots & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & | & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & | & \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} \overbrace{\frac{\partial G}{\partial s} \;\vdots\; \frac{\partial G}{\partial t}}^{\text{Velocity vectors}} \end{bmatrix}$$

Velocity vectors are both tangent to surface

we can form infinitesimal area element by considering . . .

$$\left|\frac{\partial G}{\partial s}\,ds \times \frac{\partial G}{\partial t}\,dt\right| \quad (\text{area of } \| \text{-a-gram})$$

$$= \begin{bmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\end{bmatrix} dx \times \begin{bmatrix}\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}\end{bmatrix} dt = \begin{bmatrix}dx & dy & dz\end{bmatrix}_s \times \begin{bmatrix}dx & dy & dz\end{bmatrix}_t$$

$$\boxed{d\sigma = \left|\frac{\partial G}{\partial s} \times \frac{\partial G}{\partial t}\right| ds\, dt}$$

$$\boxed{SA = \underset{\substack{\text{Surface} \\ \text{given by } G}}{\int} d\sigma = \underset{\substack{\text{parameter space} \\ \text{(st plane).}}}{\iint} \left|\frac{\partial G}{\partial s} \times \frac{\partial G}{\partial t}\right| ds\, dt}$$

EX Find SA of cone $x^2+y^2=z^2$ for $0\le z\le 2$.

To evaluate Surf. Integrals:

$$G\begin{bmatrix} s \\ t \end{bmatrix}=\begin{bmatrix} s \\ t \\ z(s,t) \end{bmatrix} \Rightarrow G\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ y \\ z(x,y) \end{bmatrix}$$

① parametrize surface

② use $d\sigma=\left|\frac{\partial G}{\partial s}\times\frac{\partial G}{\partial t}\right|ds\,dt$ $\quad\left(d\sigma=\left(1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right)^{1/2}dx\,dy\right)$

③ consider bounds

① Let's try cylindrical coords.

$x=r\cos\theta \qquad z=z$

$y=r\sin\theta$

cone: $x^2+y^2=z^2$

$r^2=z^2$

$\underline{r=z}$ $\longrightarrow$ So, surface is parametrized by

$\underline{x=z\cos\theta}$

$\underline{y=z\sin\theta}$

$\underline{z=z}$

$$G\begin{bmatrix} z \\ \theta \end{bmatrix}=\begin{bmatrix} z\cos\theta \\ z\sin\theta \\ z \end{bmatrix}$$

cycle through (z,θ) and you'll get a cone!

② $$[DG]=\begin{bmatrix} \cos\theta & -z\sin\theta \\ \sin\theta & z\cos\theta \\ 1 & 0 \end{bmatrix}$$

$$\frac{\partial G}{\partial z}\times\frac{\partial G}{\partial \theta}=\begin{vmatrix} \hat{i} & \cos\theta & -z\sin\theta \\ \hat{j} & \sin\theta & z\cos\theta \\ \hat{k} & 1 & 0 \end{vmatrix}=-z\cos\theta\,\hat{i}+z\sin\theta\,\hat{j}+z\hat{k}$$

$$\left\|\frac{\partial G}{\partial z}\times\frac{\partial G}{\partial \theta}\right\|=z\sqrt{2}$$

$\therefore\ d\sigma=z\sqrt{2}\,dz\,d\theta$

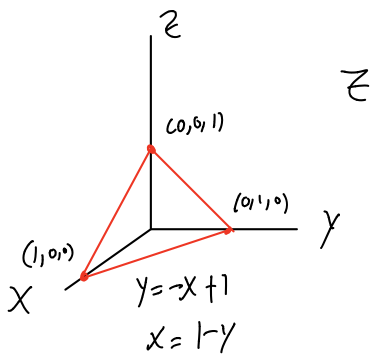
③ Bounds: $0\le z\le 2$

$0\le\theta\le 2\pi$

$$\int_0^{2\pi}\int_0^{2}z\sqrt{2}\,dz\,d\theta=\boxed{4\pi\sqrt{2}}$$

Note, cap at $z=2$ is Not Included!

Ex Find the Surface area of the plane $x+y+z=1$ for $(x,y,z) \geq 0$.



$$z = 1 - x - y$$

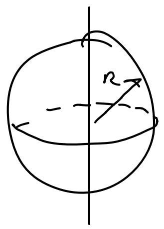
$$G\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1-x-y \end{bmatrix} \quad \begin{matrix} 0 \leq x \leq 1-y \\ 0 \leq y \leq 1 \end{matrix}$$

$$[DG] = \begin{array}{c} \begin{matrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \end{array} ; \quad d\sigma = \left\| \frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} \right\|$$

$$\begin{vmatrix} \hat{i} & 1 & 0 \\ \hat{j} & 0 & 1 \\ \hat{k} & -1 & -1 \end{vmatrix} = 1\hat{i} - (-1)\hat{j} + (1)\hat{k} \Rightarrow \sqrt{3}$$

$$\int d\sigma = \iint \left\| \frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} \right\| dx\,dy = \sqrt{3} \int_0^1 \int_0^{1-y} dx\,dy = \boxed{\frac{\sqrt{3}}{2}}$$

Ex prove that the SA of a sphere is $4\pi R^2$



Use spherical coordinates $\nearrow$ $\begin{matrix} 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{matrix}$

$$G\begin{bmatrix} \phi \\ \theta \end{bmatrix} = \begin{bmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{bmatrix} ; \quad [DG] = \begin{array}{c} \begin{matrix} \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \end{matrix} \\ \begin{bmatrix} R\cos\phi\cos\theta & -R\sin\phi\sin\theta \\ R\cos\phi\sin\theta & R\sin\phi\cos\theta \\ -R\sin\phi & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & R\cos\phi\cos\theta & -R\sin\phi\sin\theta \\ \hat{j} & R\cos\phi\sin\theta & R\sin\phi\cos\theta \\ \hat{k} & -R\sin\phi & 0 \end{vmatrix}$$

$$= (R^2\sin^2\phi\cos\theta)\hat{i} - (-R^2\sin^2\phi\sin\theta)\hat{j}$$

$$+ (R^2 \underbrace{\sin\phi\cos\phi\cos^2\theta + R^2\sin\phi\cos\phi\sin^2\theta}_{R^2\sin\phi\cos\phi})\hat{k}$$

$$|\text{cross prod}| \Longrightarrow (R^4\sin^4\phi\cos^2\theta + R^4\sin^4\phi\sin^2\theta + R^4\sin^2\phi\cos^2\phi)^{1/2}$$

$$= (R^4[\sin^4\phi + \sin^2\phi\cos^2\phi])^{1/2}; \quad \sin^2 + \cos^2 = 1$$

$$\sin^4 + \cos^2\sin^2 = \sin^2$$

$$= R^2\sin\phi.$$

$$\int d\sigma = R^2 \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= R^2 \int_0^{2\pi} -\cos\phi \Big]_0^{\pi} d\theta$$

$$= 2\pi R^2 [1+1]$$

$$= \boxed{4\pi R^2}$$

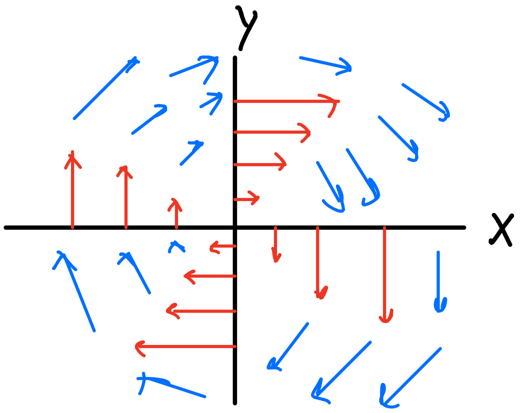
Woo.

Volume 4

(Ch 1): Fields

Vector fields assign a vector to every point in Space

Ex | Sketch the field $\vec{F}(x,y) = y\hat{i} - x\hat{j}$



Consider $x=0$, $y=0$

Sense of rotation

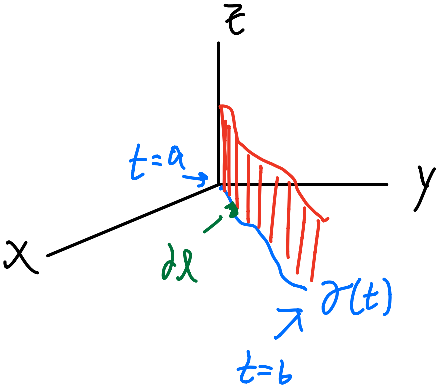
Scalar

(Ch 2): Path Integrals

For a function $f(x,y)$ and a path in the plane

$\int_{path} f\,d\ell$ = Area of curvy wall

arc length element



- How to actually compute?
- Need to parametrize integrating path → use $\gamma(t)$!

$$\boxed{\int_{path=\gamma} f\,d\ell = \int_{t=a}^{t=b} f(\gamma(t)) \left|\frac{d\gamma}{dt}\right| dt}$$

$$\left(= \int f(x(t),y(t))\,|d\gamma| \right)$$

Ex Compute the $\overset{\text{Scalar}}{\text{path}}$ integral of $f(x,y) = x^2+y^2$ over the path $\gamma(t) = \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$; $0 \le t \le 2\pi$.

$\sin(-t)$

$-\cos(-t)$

$\sin^2+\cos^2=1$

$$\int f\,d\ell = \int_{t=a}^{t=b} f(\gamma(t))\left|\frac{d\gamma}{dt}\right| dt$$

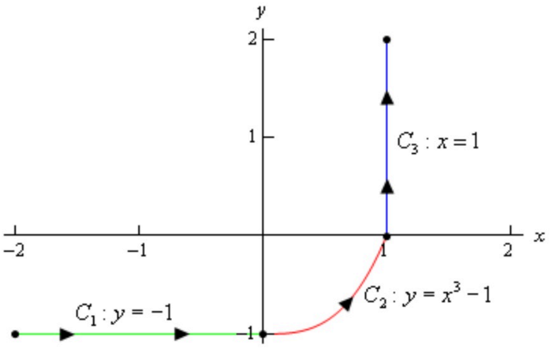
$$f(\gamma(t)) = f(x(t),y(t)) = \cos^2 t + \sin^2 t = 1$$

$$\left|\frac{d\gamma}{dt}\right| = \text{magnitude}\left(\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}\right) = (\cos^2 t+\sin^2 t)^{1/2} = 1$$

$$\rightarrow \int_{t=0}^{2\pi} 1\,dt = \boxed{2\pi}$$

Ex Evaluate $\int_C x\,d\ell$ where C is the path shown below: Hint $\int_0^1 t(1+9t^4)^{1/2}\,dt = 0.94$.

$$\boxed{\int_\gamma f\,d\ell = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt}$$



$$\int_C x\,d\ell = \int_{C_1} x\,d\ell + \int_{C_2} x\,d\ell + \int_{C_3} \overset{x=1}{\nearrow} d\ell$$

$\Rightarrow \int_{C_1} x\,d\ell$; $\gamma_1(t) = \begin{bmatrix}t\\ -1\end{bmatrix}$; $\gamma_1' = \begin{bmatrix}1\\ 0\end{bmatrix} \rightarrow |\gamma'| = 1$

$-2 \le t \le 0$

$$\int_{C_1} x\,d\ell = \int_{-2}^{0} (t)(1)\,dt = \frac{1}{2}\left[t^2\right]_{-2}^{0} = \frac{1}{2}[0-4] = \underline{-2}$$

$\Rightarrow \int_{C_2} x\,d\ell$; $\gamma_2(t) = \begin{bmatrix}t\\ t^3-1\end{bmatrix}$; $\gamma_2'(t) = \begin{bmatrix}1\\ 3t^2\end{bmatrix} \rightarrow |\gamma_2'(t)| = (1+9t^4)^{1/2}$

$0 \le t \le 1$

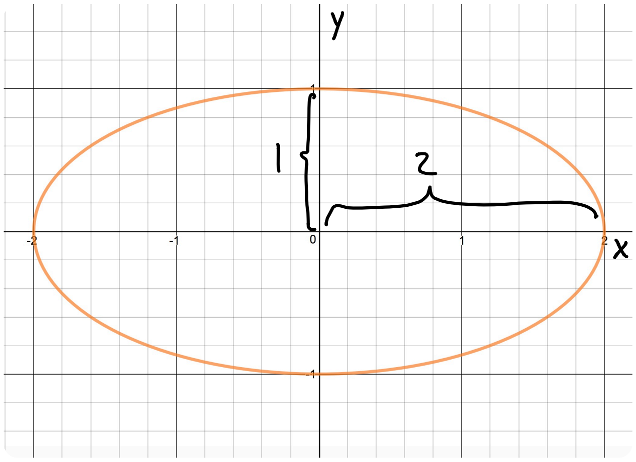
$\int^{1}$

$$\int_{C_2} x\,d\ell = \int_0 t(1+9t^4)^{1/2}\,dt = \underline{0.94}$$

$$\Longrightarrow \int_{C_3} d\ell = \underline{2} \quad \text{(look at picture!)}$$

$$\Longrightarrow \int x\,d\ell = -2+2+0.94 = \boxed{0.94}$$

Ex Setup $\int_\gamma y^2\,d\ell$ where γ is the ellipse shown traversed counterclockwise for increasing t.



$$\gamma(t) = \begin{bmatrix} 2\cos(t) \\ \sin(t) \end{bmatrix} \quad 0 \le t \le 2\pi.$$

$$\gamma'(t) = \begin{bmatrix} -2\sin t \\ \cos t \end{bmatrix} \rightarrow 4\sin^2 t + \cos^2 t$$

$$\int_\gamma y^2\,d\ell = \int_0^{2\pi} \sin^2(t)\left[4\sin^2 t + \cos^2 t\right] dt = \boxed{\int_0^{2\pi} (4\sin^4 t + \cos^2 t \sin^2 t)\,dt}$$

Now, suppose γ was the same path but traversed clockwise for increasing t. What is the parametrization now?

$$\gamma_2(t) = \begin{bmatrix} 2\cos(-t) \\ \sin(-t) \end{bmatrix} \quad 0 \le t \le 2\pi$$

(b) How does the integral $\int f\,d\ell$ over this new parametrized path compare to the value of the integral in part (a)?

It's the same: $\int_\gamma f\,d\ell = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt = \int_a^b f(\gamma(-t))\,|\gamma'(-t)|\,dt$

Will accumulate same exact values, now just in opposite order (backwards). START POINT, END PT SAME AS BEFORE

Path integrals are independent of parametrized path!!

(c) What if you used the path $-\gamma$?

How does your answer compare to part (a) and (b)?

Negative ans (a) $\longrightarrow - \int_0^{2\pi} (4\sin^4 t + \cos^2 t \sin^2 t)\,dt$

Same path but endpts reversed:

$$\int_\gamma = \int_{t_0}^{t_1} \to \int_{-\gamma} = \int_{t_1}^{t_0} = -\int_\gamma .$$

(Ch 3) : 1 Forms!

__1-forms__ : Very much related to path integrals, but a different perspective.
That's what differential forms are all about, a different perspective.

Start this discussion by considering Vector Path Integrals

What lies ahead...

Warning : beautiful math incoming

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$dF = 0$$

$$dM = J$$

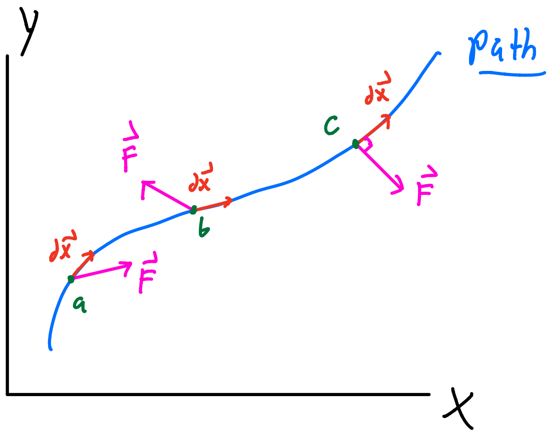
Suppose $\vec{F} = F_1(\vec{x})\,\hat{i} + F_2(\vec{x})\,\hat{j} + \dots$

and $d\vec{x} = dx_1\,\hat{i} + d\vec{x}_2\,\hat{j} + \dots$

$\to \vec{F} \cdot d\vec{x} = F_1(\vec{x})\,dx_1 + F_2(\vec{x})\,dx_2 + \dots$

$$\int_{Path} \vec{F} \cdot d\vec{x} = \int_{Path} F_1(\vec{x})\,d\vec{x}_1 + F_2(\vec{x})\,dx_2 + \dots$$

__Picture__ : Suppose we have $\vec{F}(x,y)$ in $\mathbb{R}^2$



$\int_{Path} \vec{F} \cdot d\vec{x}$ adds up whole bunch of dot products along the path!

Rank contribution to integral from $G \longrightarrow L$: G _a_ _c_ _b_ L

__Definition__ : $\vec{F} \cdot d\vec{x} = \alpha_{\vec{F}} = F_1(\vec{x})\,dx_1 + F_2(\vec{x})\,dx_2 + \dots$

Powerful! OK. $dx_1, dx_2 \dots$ correspond to $\Delta x_1, \Delta x_2, \dots$

For intuition: Suppose $\alpha = mg\,dz$; m, g const.

$du = \alpha = mg\,dz$

1) How much is Δu (not du) if I move up +3 units in z?

$$\Delta u = mg(3) = \underline{3mg}$$

2) How much is Δu if I move +100 units in x

$$\Delta u = mg(0) = \underline{0}$$

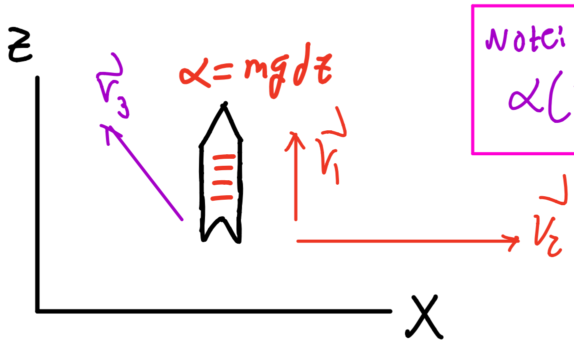
Observe: Certain directions of motion give large Δu, others do not.

Notation:

1) $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, $\alpha(\vec{v}_1) = mg(3) = 3mg$.

2) $\vec{v}_2 = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$, $\alpha(\vec{v}_2) = mg(0) = 0$.

Notice $\int \alpha = \int du = u$; $\underline{\alpha(\vec{v}) \text{ approximates integrals!}}$



Note:
$\alpha(\vec{v}_3) > \alpha(\vec{v}_1)$

$\alpha(\vec{v})$ approximates value of $\int \alpha$ when moving in dir of $\vec{v}$!!!!

Ex $\alpha = y^2dx - x^2dy$ @ $(2,-1)$ on $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$$\alpha_{(2,-1)}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = (1)^2(3) - (4)(-4) = \boxed{19}$$

Throw back to $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \rightarrow \Delta x = 3,\ \Delta y = 2$

$$\Delta f \approx \frac{\partial f}{\partial x}(3) + \frac{\partial f}{\partial y}(2) \ldots$$

Integrating 1-forms over a path: $\gamma(t)$; $a \le t \le b$.

$$\int_\gamma \alpha = \int_{t=a}^{b} \alpha_{\gamma(t)}(\gamma'(t))\,dt \longrightarrow \text{moving in dir of } \gamma \text{ (velocity)}.$$

$(x,y,z) \rightarrow (x(t),y(t),z(t))$

$$\left(\int \vec{F}\cdot d\vec{r} = \int \vec{F}(\gamma(t))\cdot \frac{d\gamma}{dt}\,dt\right) : \left(\text{Note line integrals do } \int_\gamma \vec{F}\cdot\vec{T}\,d\ell\right)$$

Ex Compute 1-form integral $\int \alpha = \int y^2dx - x^2dy$ over γ = path along $y = 4 - x^2$ from $x = -2$ to $x = 2$

$$\int_\gamma \alpha = \int_{t=a}^{b} \alpha_{\gamma(t)}(\gamma'(t))\,dt$$

Parametrize: $\begin{matrix} x = x \\ y = 4 - x^2 \end{matrix}$; $\gamma(x) = \begin{bmatrix} x \\ 4-x^2 \end{bmatrix}$ $-2 \le x \le 2$

$\gamma'(x) = \begin{bmatrix} 1 \\ -2x \end{bmatrix}$ $\left(= \begin{bmatrix} dx \\ dy \end{bmatrix}\right)$

$$\int_\gamma \alpha = \int_{-2}^{2} \left((4-x^2)^2(1) - x^2(-2x)\right)dx = \boxed{\frac{512}{15}}$$

Ex | $f(x,y) = 3x + y$. Compute the gradient 1 form of f and integrate it over the unit circle in the CCW direction.

$$df = 3dx + 1dy \, ; \quad \gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \quad 0 \leq t \leq 2\pi \rightarrow \gamma'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

$$\int_\gamma df = \int_\gamma df_{\gamma(t)}(\gamma'(t))\,dt = \int_0^{2\pi} (-3\sin t + \cos t)\,dt$$

$$= \Big[+3\cos t + \sin t\Big]_0^{2\pi} = \Big[3\cos(2\pi) - 3\cos(0) + \sin(2\pi) - \sin(0)\Big] = \boxed{0}$$

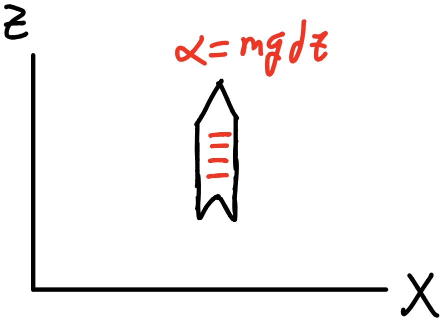
Interesting...

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)) \, ; \quad a = b$$

$\Longrightarrow 0$. more to come!

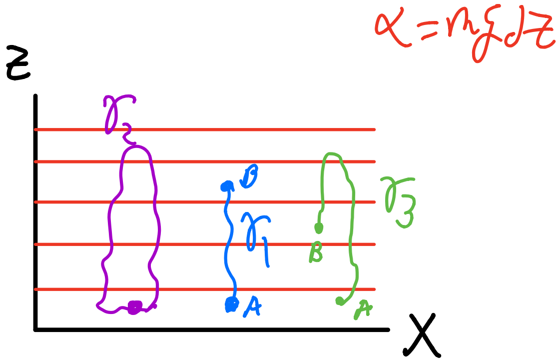
Integrating 1-forms... Intuition!

Let's use $\alpha = mg\,dz$ again



One way to draw it

or



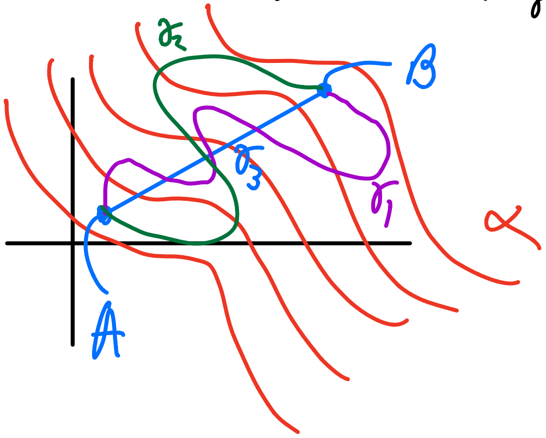
Larger value of m, lines become big (α gets bigger)

$$\boxed{\int_{\gamma_1} \alpha = 3 \, ; \quad \int_{\gamma_2} \alpha = 0 \, ; \quad \int_{\gamma_3} \alpha = 2}$$

Integrals of one forms count # signed surfaces you pass through.

(Ch 4) Independence of Path

Idea: If α is a gradient 1-form, $\int_\gamma \alpha$ depends only on the end points...



Recall a gradient 1-form is when

we take $\nabla f \cdot d\underline{x} = \underbrace{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \dots}_{\text{gradient 1-form.}}$

$= df$!!!!

$$\boxed{\int_\gamma \nabla f \cdot d\underline{x} = \int_\gamma \alpha_{\nabla f} = \int_\gamma df = f(B) - f(A).}$$

Ex Compute $\int_\gamma \nabla(e^x \cos y + xyz + \frac{1}{2}z^2) \cdot d\underline{x}$

Where γ is the straight line path from $(0,0,0) \to (0,0,1)$.

$$\alpha = \nabla(\overbrace{e^x \cos y + xyz + \tfrac{1}{2}z^2}^{f}) \cdot d\underline{x} = df$$

$$\int_\gamma \alpha = \int_\gamma df = f(END) - f(START) = f(0,0,1) - f(0,0,0)$$

$$= 1 + \tfrac{1}{2} - 1$$

$$= \boxed{1/2}$$

Ex If $\alpha = (-2x - ye^{2z}\sin(xy))\,dx - xe^{2z}\sin(xy)\,dy$
$+ (2e^{2z}\cos(xy) + \cos z)\,dz$, what is $\int_\gamma \alpha$
if γ is the circle of radius R in the plane $x+y+z=1$ centered at $(x,y,z)=(0,0,1)$?

$\int_\gamma \alpha = ?$ Well, if α is grad 1-form, $\int_\gamma \alpha = f(\text{END}) - f(\text{START})$

1) $\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$

2) $\frac{\partial}{\partial z}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial z}$

3) $\frac{\partial}{\partial z}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial z}$.

1) $-ye^{2z}\cos(xy)\cdot x - e^{2z}\sin(xy)$
$= -xe^{2z}\cos(xy)y - e^{2z}\sin(xy)$ ✓

2) $-2xe^{2z}\sin(xy) = -2e^{2z}\sin(xy)\cdot x$ ✓

3) $-2ye^{2z}\sin(xy) = -2e^{2z}\sin(xy)\cdot y$ ✓

α is a grad 1-form!

$\therefore \int_\gamma \alpha = f(\text{END}) - f(\text{START})$; START = END for circle

$= f(\text{END}) - f(\text{END}) = \boxed{0}$

Ex (a) Find $\int_\gamma 2dx - 2ydy$ where γ is the

path parametrized by $\gamma(t) = \begin{bmatrix} \sin(e^t) \\ \cos(e^t) \end{bmatrix}$ $0 \le t \le 2$.

Is α grad 1-form? $\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$

$0 = 0$ ✓ yes

Find the potential function of $\alpha = df = 2dx - 2y\,dy$ for $f: \mathbb{R}^2 \to \mathbb{R}$.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = 2 \qquad \frac{\partial f}{\partial y} = -2y$$

$$\partial f = 2dx$$

$$f(x,y) = 2x + g(y) \longrightarrow f(x,y) = 2x - y^2$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -2y$$

$$dg = -2y\,dy$$

$$g(y) = -y^2 + c.$$

$$\rightarrow f(END) - f(START)$$

$$= f(\sin(e^2), \cos(e^2)) - f(\sin(e^0), \cos(e^0))$$

$$= f(\sin(e^2), \cos(e^2)) - f(\sin(1), \cos(1))$$

$$= 2\sin(e^2) - \cos(e^2) - (2\sin(1) - \cos^2(1))$$

$$= \boxed{2\sin(e^2) - \cos(e^2) - 2\sin(1) + \cos^2(1)}$$

(Ch 5) Work, Circulation, flux

Work: Just a path integral!

$$W = \int_\gamma \vec{F} \cdot d\underline{x} = \int_\gamma (F_x, F_y, F_z) \cdot (dx, dy, dz) = \int_\gamma F_x dx + F_y dy + F_z dz = \int_\gamma \alpha_{\vec{F}}.$$

Ex Find the work done by the force field $\vec{F} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in moving an object along the path

$$\gamma(t) = \begin{bmatrix} \cos(\pi t) \\ t^2 \\ \sin(\pi t) \end{bmatrix} \quad 0 \le t \le 1.$$

$$W_\gamma = \int_\gamma \vec{F} \cdot d\underline{x} = \int_\gamma \alpha_{\vec{F}} = \int_{t_0}^{t_1} \alpha_{\gamma(t)}(\gamma'(t))\,dt$$

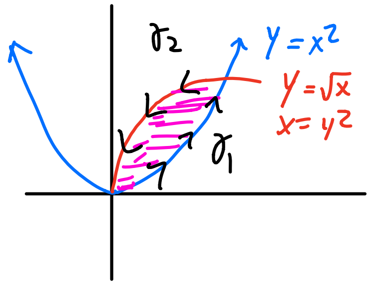
$$\rightarrow \alpha = x\,dx + y\,dy + z\,dz \; ; \quad \gamma'(t) = \begin{bmatrix} -\pi\sin(\pi t) \\ 2t \\ \pi\cos(\pi t) \end{bmatrix}$$

$$W_\gamma = \int_{t=0}^{1}\left(\cancel{\cos(\pi t)(-\pi\sin(\pi t))} + t^2(2t) + \cancel{\sin(\pi t)(\pi\cos(\pi t))}\right)dt$$

$$= 2\int_0^1 t^3\,dt = \frac{2}{4} = \boxed{\frac{1}{2}}$$

Circulation: It's just path integral (or work done) over a closed loop!

Ex Compute CCW circ. in plane of $\vec{F} = -2y\hat{i} + 5x\hat{j}$ along the region determined by $x \geq y^2$; $y \geq x^2$. Don't use greens thm to start Jake



$x = y^2$

$y = \sqrt{x}$

$$\int_\gamma \alpha_F = \int_{\gamma_1}\alpha_{\vec{F}} + \int_{\gamma_2}\alpha_{\vec{F}}$$

$$\alpha = -2y\,dx + 5x\,dy$$

$$\gamma_1(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix} \quad \gamma_1'(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix} \quad t: 0 \to 1$$

$$\gamma_2(t) = \begin{bmatrix} t^2 \\ t \end{bmatrix} \quad \gamma_2'(t) = \begin{bmatrix} 2t \\ 1 \end{bmatrix} \quad t: 1 \to 0$$

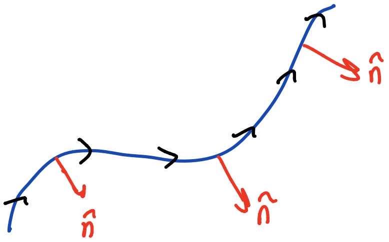
$$\int_{\gamma_1}\alpha_F = \int_0^1 \left(-2(t^2)(1) + 5(t)(2t)\right)dt$$

$$= \int_0^1 \left(-2t^2 + 10t^2\right)dt = \frac{8}{3}$$

$$\int_{\gamma_2}\alpha_F = \int_1^0 \left(-2t(2t) + 5(t^2)(1)\right)dt = \int_1^0 \left(-4t^2 + 5t^2\right)dt = -\frac{1}{3}$$

$$\therefore \int_\gamma \alpha = \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}$$

Flux: Suppose you have a curve with unit normal $\hat{n}$:

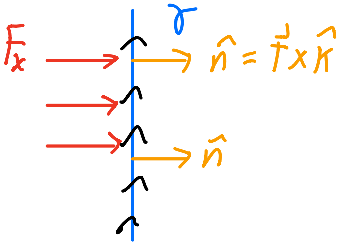


Convention: dir($\hat{n}$) determined by
$\vec{T} \times \hat{K}$ (= unit tangent $\times \hat{k}$).

Let $\vec{F} = F_x \hat{i} + F_y \hat{j}$

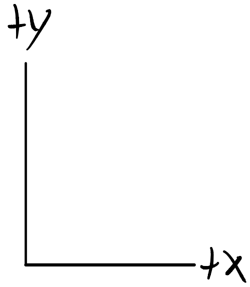
$$\boxed{\text{Flux of } \vec{F} \text{ across } \gamma = \phi_\gamma = \int_\gamma \phi_{\vec{F}} = \int F_x\,dy - F_y\,dx}$$

Geometry argument for why: If γ is a vertical line

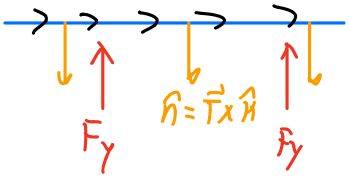


What matters is x comp. flux

$$\phi_{\vec{F}} \text{ here} = \int_\gamma \vec{F} \cdot \hat{n}\,ds = \int_\gamma F_x\,dy$$



If γ is a horiz line...



What matters is the $-y$ component of flux

$$\phi_{\vec{F}} \text{ here} = \int_\gamma \vec{F} \cdot \hat{n}\,ds = \int_\gamma -F_y\,dx$$

In plane, $\vec{F} = F_x \hat{i} + F_y \hat{j}$ so

$$\phi_{\vec{F}} = \phi(F_x) + \phi(F_y) = \int_\gamma F_x\,dy - F_y\,dx$$

More "mathy" Proof:

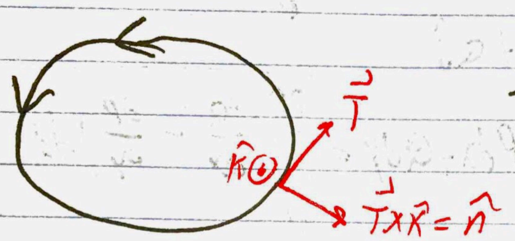
Aside: Flux formula... Where does it come from?

$$\Phi_\gamma = \int_\gamma \vec{F}\cdot\hat{n}\,ds$$ You have to accept this first.

arc length element

unit normal : direction : $\hat{n} = \vec{T}\times\hat{k}$

unit tangent vector



$$\vec{T} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}$$

$$\hat{n} = \vec{T}\times\hat{k} = \left(\frac{\partial x}{\partial s}\hat{i} + \frac{\partial y}{\partial s}\hat{j}\right)\times\hat{k}$$

$$\underline{\hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}}$$

$$\to \vec{F} = F_x\hat{i} + F_y\hat{j}$$

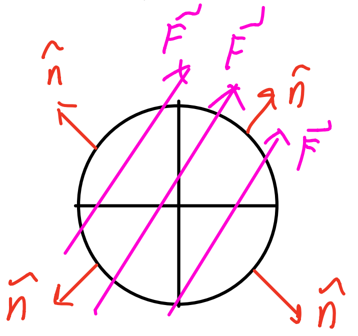
$$\vec{F}\cdot\hat{n} = F_x\frac{\partial y}{\partial s} - F_y\frac{\partial x}{\partial s}$$

$$\varphi_\gamma = \int_\gamma \vec{F}\cdot\hat{n}\,ds = \int\left(F_x\frac{dy}{ds} - F_y\frac{dx}{ds}\right)ds = \int_\gamma F_x\,dy - F_y\,dx$$ ①

Ex | Compute the flux of $\vec{F}(x,y) = 3\hat{i} + 5\hat{j}$ through the circle $x^2+y^2=1$ with positive flux in the direction of the outward unit normal

What do you expect answer to be?? (0).

$\Phi_F = F_x\,dy - F_y\,dx$; assumes outward unit normal when path is traversed CCW (in derivation $\hat{n} = \vec{T}\times\hat{k}$ is outward normal for CCW traversal).



$$\text{Flux}(\vec{F}) = \int_\gamma \Phi_F = \int_{\substack{\text{circle}\\ \text{(unit)}}} 3\,dy - 5\,dX$$

$$\gamma(t) = \begin{bmatrix}\cos t\\ \sin t\end{bmatrix} ; \gamma'(t) = \begin{bmatrix}-\sin t\\ \cos t\end{bmatrix} \qquad 0 \le t \le 2\pi$$

$$\text{Flux}(\vec{F}) = \int_0^{2\pi} \left(3\cos t + 5\sin t\right) dt$$

$$= 3\int_0^{2\pi} \cos t \, dt + 5\int_0^{2\pi} \sin t \, dt$$

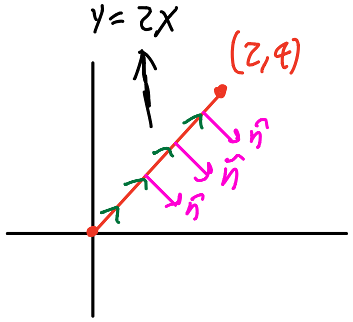
$$= 3\left[-\sin t\right]_0^{2\pi} + 5\left[-\cos t\right]_0^{2\pi}$$

$$= 5\left[-1-(-1)\right]$$

$$= \boxed{0}$$

As expected.

Ex Find the flux of the vector field $\vec{F} = 2y\hat{i} - 4x\hat{j}$ along the straight line path from $(0,0) \to (2,4)$ with positive flux in the direction of the downwards unit normal to the line.



$$\int_\gamma \varphi_F = \int_\gamma F_x \, dy - F_y \, dx$$

$$\gamma(t) = \begin{bmatrix} t \\ 2t \end{bmatrix} \qquad 0 \le t \le 2$$

$$\gamma'(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\int_\gamma \varphi_F = \int_0^2 \left(2(2t)(2) - (-4t)(1)\right) dt$$

$$= \int_0^2 (8t + 4t)\, dt = 12\int_0^2 t \, dt = 6t^2\Big|_0^2$$

$$= 6 \cdot 4 = \boxed{24}$$

(Ch 6) Green's Thm: Nice Geometrical Derivation posted on TA site

For a vector field $\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$

$$\boxed{\text{Circulation along } \underbrace{\text{bdry of } D}_{\partial D} = \int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

We also have $\text{Flux}(\vec{F}) = \int_\gamma \phi_F = \int_\gamma P\,dy - Q\,dx$; $\vec{F} = P\hat{i} + Q\hat{j}$

Why? Suppose we have $\vec{G} = -Q\,\hat{i} + P\,\hat{j}$

$$\text{Circulation}(\vec{G}) = \int_\gamma -Q\,dx + P\,dy = \int_\gamma P\,dy - Q\,dx = \text{Flux}(\vec{F})$$

OH! $\text{circ}(\vec{G}) = \int_{\partial D} P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right)\right) dA$

$$\text{Flux}(\vec{F}) = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

must be $\frac{\text{circ}}{\text{Area}}$

Summarize: $\vec{F} = P\,\hat{i} + Q\,\hat{j}$

$$\boxed{\begin{aligned} \text{Circ}(\vec{F}) &= \int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ \text{Flux}(\vec{F}) &= \int_{\partial D} P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \end{aligned}}$$

For flux, Change is what

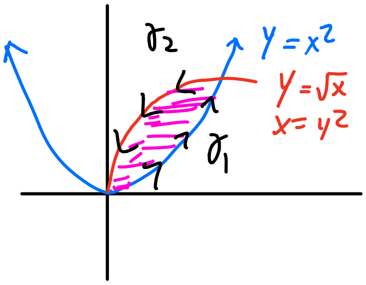
must be $\frac{\text{flux}}{\text{Area}}$

matters (no change, flux thru surface=0)

Ex Same as before: Compute CCW circ. in the plane of $\vec{F}=-2y\hat{i}+5x\hat{j}$ along the region determined by $x\geq y^2, y\geq x^2$. Use Green's!!

$$\text{circ}(\vec{F}) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA; \qquad P=-2y \quad Q=5x$$

$$\frac{\partial P}{\partial y} = -2 \qquad \frac{\partial Q}{\partial x} = 5$$



$$\Rightarrow \int_0^1 \int_{x^2}^{\sqrt{x}} 7 \, dy\, dx = 7\int_0^1 (\sqrt{x}-x^2)\, dx$$

$$= 7\left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= 7\left[\frac{2}{3} - \frac{1}{3}\right] = \boxed{\frac{7}{3}}$$

Easier than before!

Ex Find flux of $\vec{F} = y\hat{i} - x\hat{j}$ out of each side of the dodecagon (a 12 sided polygon) with positive flux in the direction of the outward unit normal.

$$\int_\gamma \phi_F = \int_{\gamma_1} \phi_F + \int_{\gamma_2} \phi_F + \cdots + \int_{\gamma_{12}} \phi_F$$

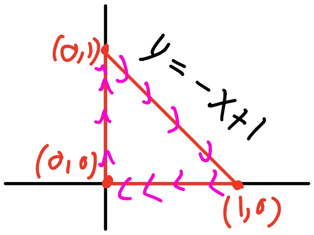
OR Green's Thm! It assumes outward unit normals so we're good.

$$\text{Flux}(\vec{F}) = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA; \qquad P=y \quad Q=-x$$

$$\frac{\partial P}{\partial x} = 0 \qquad \frac{\partial Q}{\partial y} = 0$$

$$\therefore \text{Flux}(\vec{F}) = \iint_D 0 \, dA = \boxed{0}$$

Ex Compute the clockwise circulation of the vector field $\vec{F} = x^4\hat{i} + xy\hat{j}$ around the triangle w/ vertices (0,0), (1,0), (0,1).



Easy enough to compute $\int_\gamma \alpha$ (line integral) but let's use Green's.

Note! Green's thrm assume CW circulation. But CW circ = -CCW circ ($\int \vec{F}\cdot d\vec{r} = -\int \vec{F}\cdot -d\vec{r}$)

So:

$$CW\ circ(\vec{F}) = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$Q = xy \qquad P = x^4$

$\frac{\partial Q}{\partial x} = y \qquad \frac{\partial P}{\partial y} = 0 \to \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y$

$$-\int_0^1 \int_0^{-x+1} y \, dy\, dx = \boxed{-\frac{1}{6}}$$

(Ch 7) Grad-Curl-div

$$\text{Idea: circulation}(\vec{F}) = \iint_D \left(\overbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}^{\text{must be cir/gren}}\right) dA$$

(3d add up along surface, need vec form)

$$\text{Flux}(\vec{F}) = \iint_D \left(\underbrace{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}_{\text{must be Flux/gren}}\right) dA$$

$\Bigg\}\ \vec{F} = P\hat{i} + Q\hat{j}$

Define $\nabla \times \vec{F}$ = Circulation densities in each plane ($\hat{i}$ tells us YZ circ, $\hat{j}$ tells us XZ circ, $\hat{k}$ tells us XY circ).

$\nabla \cdot \vec{F}$ = Flux density (scalar)

↳ (Spatial → 3D we add up inside vol)

and $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

EX $\vec{F} = xy^2\hat{i} + \sin(xy)\hat{j} + z\hat{k}$

Compute $\text{div}(\vec{F})$

$$\boxed{\nabla \cdot \vec{F} = y^2 + x\cos(xy) + 1}$$

Compute $\text{curl}(\vec{F})$: Interesting result.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \frac{\partial}{\partial x} & xy^2 \\ \hat{j} & \frac{\partial}{\partial y} & \sin(xy) \\ \hat{k} & \frac{\partial}{\partial z} & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial}{\partial z}\sin(xy)\right)\hat{i} - \left(\frac{\partial z}{\partial x} - \frac{\partial}{\partial z}xy^2\right)\hat{j} + \left(\frac{\partial}{\partial x}\sin(xy) - \frac{\partial}{\partial y}xy^2\right)\hat{k}$$

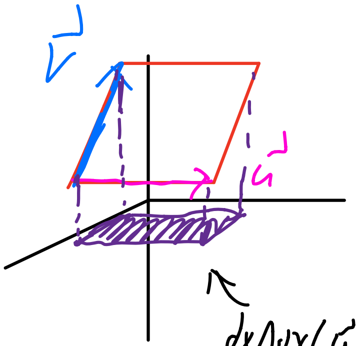
$$= \boxed{0\hat{i} + 0\hat{j} + (y\cos(xy) - 2xy)\hat{k}}$$

There is only circ. in $\hat{k}$ dir

→ only circulation ∥ to xy plane!

(Curl is z independent here).

(Ch 8) Forms in 3D: (x, y, z) Euclidian Forms



$dx \wedge dy(\vec{u}, \vec{v})$

$= + \#$ here.

Given two Vectors: $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

We know $\|\vec{u} \times \vec{v}\|$ gives area of parallelogram.

What is the Signed Projection of this area onto the xy plane?

→ Consider the matrix: $[\vec{u} \,\vdots\, \vec{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix}$

these are the components that matter for xy projection

Signed Projection onto xy plane $= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \equiv dx \wedge dy(\vec{u}, \vec{v})$.

Wedge has same sign mechanics as cross prod! (via det).

- $dx_i \wedge dx_j(\vec{u},\vec{v})$ gives signed projected area of parallelogram spanned by $\vec{u},\vec{v}$ on $x_i - x_j$ plane.
- $dx \wedge dy \wedge dz$ gives signed volume (via determinant)

properties of wedge prod:

Associative: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

Distributive: $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

Anti-Symmetric: $\alpha \wedge \beta = -\beta \wedge \alpha$

$$\to \alpha \wedge \alpha = 0$$

Ex What is the area of the projection parallelogram spanned by $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on the yz plane?

$$\text{Proj on } yz = dy \wedge dz(\vec{u},\vec{v}) = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$$

$$\boxed{\text{Projected Area} = |-1| = 1}$$

Ex Compute $(5dx - dy) \wedge (dx \wedge dy - 2dx \wedge dz)$

$$= 5\,\cancel{dx \wedge (dx \wedge dy)} + 5\,dx \wedge (\cancel{-2}\,dx \wedge dz)$$

$$- dy \wedge (\cancel{dx} \wedge dy) - dy \wedge (-2\,dx \wedge dz)$$

$$= 2\,dy \wedge dx \wedge dz = \boxed{-2\,dx \wedge dy \wedge dz}$$

Why $dy\wedge dx\wedge dy = 0$? this prod takes 3 vectors and returns projected <u>volume</u>

$$dy\wedge dx\wedge dy = \begin{vmatrix} u_y & v_y & w_y \\ u_x & v_x & w_x \\ u_y & v_y & w_y \end{vmatrix} = \begin{vmatrix} u_y & u_x & u_y \\ v_y & v_x & v_y \\ w_y & w_x & w_y \end{vmatrix}$$

same vectors!

Vol spanned by 2 vectors = 0

Now, how do we differentiate these things?

Just like df takes f from 0 form → 1 form...

$d\alpha$ takes α from k form → $k+1$ form.

(this is because $\int_{\partial D} \alpha = \int_D d\alpha$)

Ex

$\alpha = 2xy\,dx - y^4\,dz$

$d\alpha = ??$

$$d\alpha = d(2xy)\,dx - d(y^4)\,dz$$
$$= (df_1)\wedge dx - d(y^4)\wedge dz$$
$$= (2y\,dx + 2x\,dy)\wedge dx - (4y^3\,dy)\wedge dz$$
$$= 2x\,dy\wedge dx - 4y^3\,dy\wedge dz$$

$$\boxed{d\alpha = -2x\,dx\wedge dy - 4y^3\,dy\wedge dz}$$

Christ also defined flux 2 form as

$$\Phi_{\vec F} = F_x\, dy\wedge dz + F_y\, dz\wedge dx + F_z\, dx\wedge dy$$

And he showed that $d\alpha_{\vec F} = \Phi_{\nabla\times\vec F}$

and $$d\Phi_{\vec F} = (\nabla\cdot\vec F)\, dx\wedge dy\wedge dz.$$

More to come on why... Kind of.

(ch 9): Integrating 2 forms

$$\int_D f(x,y)\, \underbrace{dx\wedge dy}_{\text{Just an area form ... But it's oriented}} = \iint_D f(x,y)\, dA$$

More generally, if B is a 2 form ; $G(s,t)$ parametrizes integrating surface.

$$\int_{\text{surface}} B = \iint_{\text{parameter space}} B_{G(s,t)}\,[DG]\,ds\,dt$$

↖ gives two tangent vectors to feed B.

And this simplifies Green's Thrm...

(K comp of curl)

$$\int_{\partial D} \underbrace{F_x\, dx + F_y\, dy}_{\alpha_{\vec F}} = \iint_D \underbrace{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\, dy}_{d\alpha_{\vec F}}$$

$$\alpha_{\vec F} \longrightarrow d\alpha_{\vec F}$$

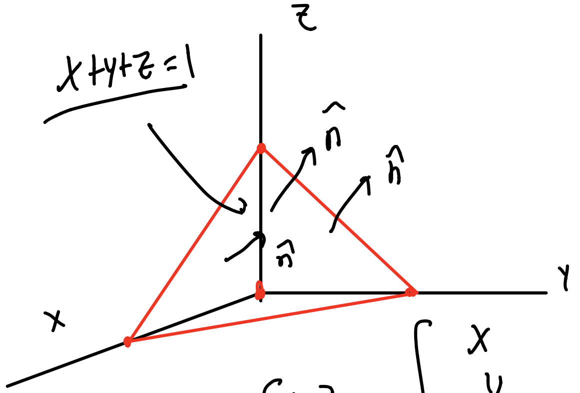
$$\boxed{\int_{\partial D} \alpha_{\vec F} = \int_D \alpha_{\vec F}}$$

Ex The flux 2-form of $\vec{F} = (F_x, F_y, F_z)$ is

$$\Phi_{\vec{F}} = F_x\, dy \wedge dz + F_y\, dz \wedge dx + F_z\, dx \wedge dy$$

Suppose $\vec{F} = 3y\,\hat{i} - 2x\,\hat{j} + 2z\,\hat{k}$

Compute the flux of $\vec{F}$ out of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Use the direction away from the origin $\perp$ to the surface as + flux.



$$\Phi_{\vec{F}} = 3y\, dy \wedge dz - 2x\, dz \wedge dx + 2z\, dx \wedge dy$$

$$\int_{\text{surf}} B = \int_{\substack{\text{Param} \\ \text{Space}}} B_{G(s,t)}([DG])\, ds\, dt$$

$$G\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1-x-y \end{bmatrix}$$

$0 \leq y \leq 1-x$

$0 \leq x \leq 1$

$$[DG] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

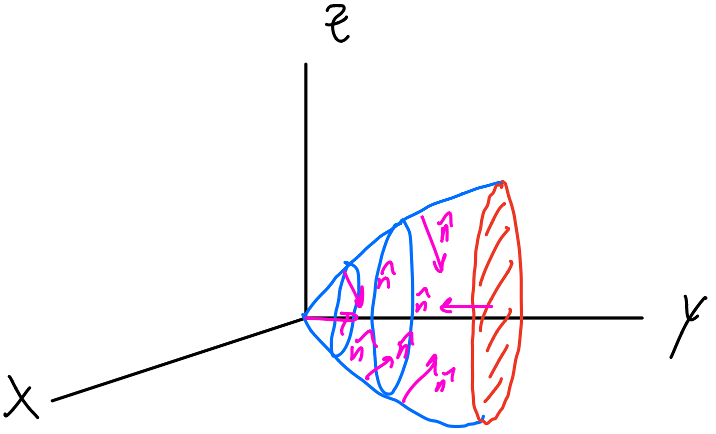
$$\int_{\text{surface}} \phi_F = \int_0^1 \int_0^{1-x} 3y \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} - 2x \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} + 2(1-x-y) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \big(3y - 2x + 2(1-x-y)\big)\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (-4x + y + 2)\, dy\, dx = \boxed{\frac{1}{2}}$$

$\boxed{\text{Ex}}$ Compute the flux of the vector field $\vec{F} = y\hat{j} - z\hat{k}$ through the paraboloid $y = x^2+z^2$, $0 \le y \le 1$ and the disk $x^2+z^2 \le 1$ at $y=1$. Use flux into the surface as positive.

Hint: $\int_0^{2\pi} (\cos^2\theta + 3\sin^2\theta)\, d\theta = 4\pi$



$$\Phi_{\vec F} = y\, dz \wedge dx - z\, dx \wedge dy$$

First find flux through paraboloid, then find flux through disk!

Let's think...

$$\int_{\text{Parab.}} \Phi_{\vec F} = \int_{\substack{s,t \\ \text{space}}} \Phi_{\vec F_{G\binom{s}{t}}} [DG]\, ds\, dz ,$$

$$G\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} s \\ s^2+t^2 \\ t \end{bmatrix} \rightarrow G\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} x \\ x^2+z^2 \\ z \end{bmatrix}$$

$$[DG] = \begin{bmatrix} 1 & 0 \\ 2x & 2z \\ 0 & 1 \end{bmatrix} \quad \underset{\vec u}{\uparrow} \quad \underset{\vec v}{\uparrow}$$

; Columns of $[DG]$ give velocity vectors tangent to surface.

We feed the 2 form these vectors to integrate over area!

- We have a couple orientations to keep track of.
  - → orientation of surface
  - → $dx\wedge dy(\vec{u},\vec{v})$ or $dx\wedge dy(\vec{v},\vec{u})$
    - ↳ making the correct choice here orients the surface. Use right hand rule.

$$\int_{\text{Parab.}} \varphi_{\vec{F}} = \int_{\text{Parab.}} y\, dz\wedge dx - z\, dx\wedge dy$$

$$dz\wedge dx(\vec{u},\vec{v}) = \begin{array}{c|cc|l} & 0 & 1 & z \text{ comp} \\ & 1 & 0 & x \text{ comp} \end{array}$$

We want this to be a positive # b/c flux coming through the xz plane in +y dir should be positive for paraboloid.

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \longrightarrow 1$$ (positive flux for $F_y > 0$, negative flux for $F_y < 0$).

or $\vec{F}\cdot -\hat{j} \to \vec{F}\cdot\hat{j}$ $F_y > 0$ + flux

$F_y < 0$ − flux

$$y\, dz\wedge dx \longrightarrow y(1)$$

now for $-z\, dx\wedge dy$; $dx\wedge dy(\vec{u},\vec{v}) = \begin{vmatrix} 1 & 0 \\ 2x & 2z \end{vmatrix}$

$z > 0$: want flux coming out xy plane in $\hat{k}$ dir. to be negative (look at picture, $n_z = -z$ direction)

For $z<0$: Want flux moving in $+\hat{k}$ ($+z$ dir) to be positive

Want $dx \wedge dy(\vec{u},\vec{v}) = \begin{vmatrix} 1 & 0 \\ 2x & 2z \end{vmatrix}$ to be <0 for $z>0$

>0 for $z<0$.

$dx \wedge dy = 2z$; $\quad z>0$ it's $+$

$z<0$ it's $-$

$2z \to -2z \quad z>0$ it's $-$

$z<0$ it's $+$

$$-z\, dx\wedge dy \to -z(-2z) = 2z^2$$

$$\Rightarrow \int_{\text{parab}} \phi_F = \int_{\text{parab}} y + 2z^2 = \iint \left((x^2+z^2)+2z^2\right) dx dz$$

$$= \iint (x^2+3z^2)\, dx\, dz$$

$x = r\cos\theta \qquad z = r\sin\theta \qquad 0 \le r \le 1$, $0 \le \theta \le 2\pi$

(Look at Paraboloid) $\quad dxdz = r dr d\theta$

$$= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta + 3r^2\sin^2\theta)\overbrace{r\,dr}\,d\theta$$

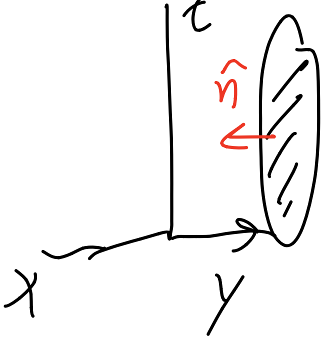
$$= \int_0^{2\pi} \left(\frac{r^4}{4}\cos^2\theta + \frac{3}{4}r^4\sin^2\theta \right) \Big|_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}(\cos^2\theta + 3\sin^2\theta)\,d\theta = \frac{4\pi}{4} = \boxed{\pi}$$

Now for disk:

$\varphi_{\vec F} = y\,dz\wedge dx - z\,dx\wedge dy$

$$\int_{\text{disk}} \phi_F = \int_{\substack{\text{disk at}\\ y=1 \text{ in}\\ xz \text{ plane}}} y\,dz\wedge dx - z\,dx\wedge dy$$



$dx\wedge dy$ for disk $= 0$

$y = 1$

$$\Rightarrow \int_{\substack{\text{unit}\\ \text{disk}}} 1\,\overbrace{dz\wedge dx}^{(\hat{j})}$$

We want flux in $-y$ direction to be >0

$\therefore$ Flux in $+y$ direction should be <0

$$-\int_{\substack{\text{unit}\\ \text{disk}}} dz\,dx = -\int_{\substack{\text{unit}\\ \text{disk}}} dA$$

$$= -\pi(1)^2$$

$$= \boxed{-\pi.}$$

$\rightarrow +\pi - \pi = \boxed{0} \rightarrow$ final answer!

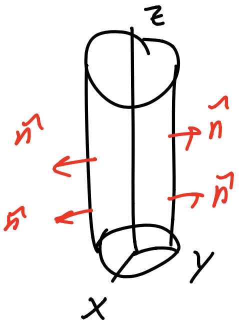
$\left(\text{Flux}_{\text{parab}} + \text{Flux}_{\text{disk}} \right)$

Faster way to do this: Look at $[DG] = \left[\frac{\partial G}{\partial s} \,\middle|\, \frac{\partial G}{\partial t} \right]$

If $\frac{\partial G}{\partial s} \times \frac{\partial G}{\partial t}$ is pointing opposite your normal vector:

$$\left.\begin{aligned} &\frac{\partial G}{\partial s} \rightarrow -\frac{\partial G}{\partial s} \\ \underline{\underline{\text{or}}}\quad &\frac{\partial G}{\partial t} \rightarrow -\frac{\partial G}{\partial t} \end{aligned}\right\} \text{not both}$$

now your parametrization has same orientation as surface. B/c $dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$ all take components of velocity vector, if cross prod of velocity vectors points in same direction as the normal, integrate without much thought.



Ex Put what we learned above to use...

Compute the the flux of $\vec{F} = \frac{1}{9} x \hat{i} + \frac{1}{9} y \hat{j} + \frac{1}{9} z \hat{k}$ out of the cylinder $x^2 + y^2 = 9$ for $0 \le z \le 5$.

$$\int_{\text{cylinder}} \Phi_{\vec{F}} = \int_{\text{cylinder}} \frac{1}{9} x \, dy \wedge dz + \frac{1}{8} y \, dz \wedge dx + \frac{1}{9} z \cancelto{0}{dx \wedge dy}$$

Projected no area in xy plane

$$G\begin{bmatrix} z \\ \theta \end{bmatrix} = \begin{bmatrix} 3\cos\theta \\ 3\sin\theta \\ z \end{bmatrix} \quad \begin{matrix} (r=3) \\ 0 \le \theta \le 2\pi \\ 0 \le z \le 5 \end{matrix} \qquad [DG] = \begin{array}{c} \begin{matrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial \theta} \end{matrix} \\ \begin{bmatrix} 0 & -3\sin\theta \\ 0 & 3\cos\theta \\ 1 & 0 \end{bmatrix} \end{array}$$

$\frac{\partial G}{\partial z} \times \frac{\partial G}{\partial \theta}$ points inwards fix this by letting

$$[DG] \to \begin{bmatrix} 0 & -3\sin\theta \\ 0 & 3\cos\theta \\ -1 & 0 \end{bmatrix}$$

$$\int_{\text{cylinder}} \Phi_{\vec{F}} = \int_{\text{cylinder}} \frac{1}{9} x \, dy \wedge dz + \frac{1}{9} y \, dz \wedge dx + \frac{1}{9} z \, \cancelto{0}{dx \wedge dy}$$

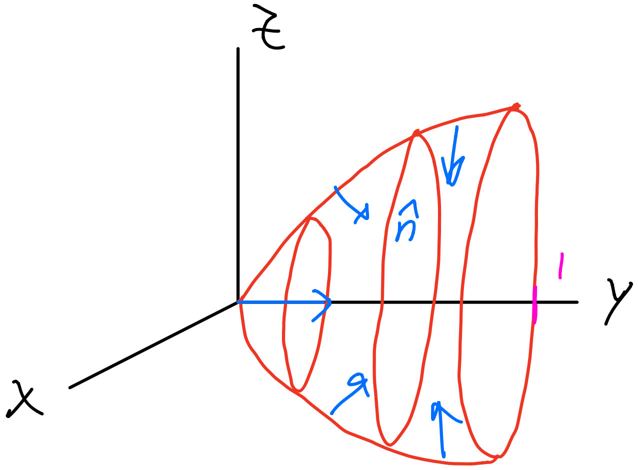
$$= \frac{1}{9} \int_0^{2\pi} \int_0^5 \left(3\cos\theta \begin{vmatrix} 0 & 3\cos\theta \\ -1 & 0 \end{vmatrix} + 3\sin\theta \begin{vmatrix} -1 & 0 \\ 0 & -3\sin\theta \end{vmatrix} \right) dz \, d\theta$$

do not use $r dr d\theta$ or $3 dz d\theta$ here. Just $dz \, d\theta$. We take care of area forms in determinants.

$$= \frac{1}{9} \int_0^{2\pi} \int_0^5 \left(9\cos^2\theta + 9\sin^2\theta \right) dz \, d\theta$$

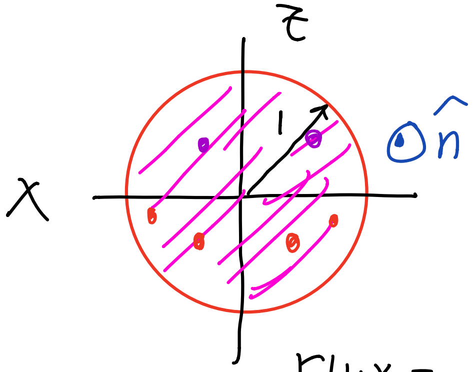
$$= \int_0^{2\pi} \int_0^5 1 \, dz \, d\theta = 5 \cdot 2\pi = \boxed{10\pi}$$

Ex Integrate the flux 2-form $\phi = (x+z)\,dz\wedge dx$ along the paraboloid $y=x^2+z^2$ for $0 \le y \le 1$. The positive direction of flux is inward (along +x-axis at origin).



$$\int_{\text{Ellipsoid}} \phi = \int_{\text{Ellipsoid}} (x+z)\, dz\wedge dx$$

$dz\wedge dx$ is signed projected area in xz plane!



$x+z$ is y comp of flux, Positive y comp $(x+z>0)$ should contribute positive flux!

$$\text{Flux} = \iint_{\substack{\text{unit}\\ \text{circle}}} (x+z)\, dz\, dx = \boxed{0}$$

Symmetry tells us

If flux was out of surface: $-\int (x+z)\, dz\, dx$.

(Ch 10): Gauss' Thrm

$$\boxed{\begin{aligned}\int_{\partial D} \varphi_{\vec F} &= \int_D d\varphi_{\vec F} \\ &= \int_D \nabla\cdot\vec F \, dV/dA\end{aligned}}$$

↑ if in plane

When to use...

When asked: "What is the flux of $\vec F$ out of the closed surface?"

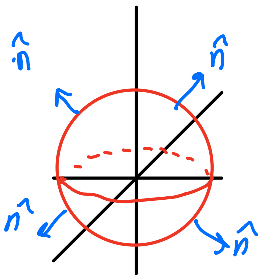
Do NOT use if asked "What is the flux of the Curl of $\vec F$?"

$\nabla\cdot\vec F$ is flux per unit volume (flux density).

Ex What is the flux of $\vec F = xy^2\,\hat i + yz^2\,\hat j + x^2 z\,\hat k$ out of the sphere of radius 3 centered at the origin?

$$\int_{\partial D} \varphi_{\vec F} = \int_D \varphi_{\vec F} = \int_D \nabla\cdot\vec F \, dV$$

↗ assumes $\hat n$ outwards



Use $\nabla\cdot\vec F$ formulation:

$$\nabla\cdot\vec F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(xy^2, yz^2, x^2 z\right)$$

$$= y^2 + z^2 + x^2 = x^2 + y^2 + z^2$$

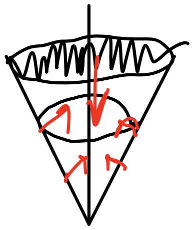
$$\text{Flux}(\vec F) = \iiint_{\substack{\text{sphere's}\\\text{Interior}}} (x^2+y^2+z^2)\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{3} (\rho^2)\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \boxed{\frac{972\pi}{5}}$$

Ex Compute the flux of the vector field $\vec{F} = (xy + \tanh(z - e^{-y}))\hat{i} + (e^{x\cos(xz/2)})\hat{j} + (\sinh^2(1-x) + x - yz)\hat{k}$ Into the cone $z = \sqrt{x^2+y^2}$ for $0 \le z \le 1$ and Into the cap of radius 1 at $z = 1$.

$$\text{Gauss: } \quad \text{Flux}(\vec{F}) = -\int_D \nabla \cdot \vec{F} \, dV$$

D: Cone interior



$$\nabla \cdot \vec{F} = y + 0 - y = \boxed{0}$$

Ex Compute the flux of the vector field $\vec{F} = \frac{1}{9}(x\hat{i} + y\hat{j} + z\hat{k})$ out of the cylinder of radius 3: $x^2+y^2=9$ for $0 \le z \le 5$. Assume the cylinder is capped off on both ends by disks of radius 3.

$$\text{Flux}(\vec{F}) = \iiint\limits_{\substack{\text{cylinder}\\\text{Interior}}} \nabla \cdot \vec{F} \, dV = \iiint\limits_{\substack{\text{cylinder}\\\text{interior}}} \frac{3}{9} \, dV = \frac{1}{3}\pi(3)^2(5)$$

$$= \boxed{15\pi}$$

(Ch 11): Stoke's Thrm

Here's all the thrms in one place:

Stokes: Flux of curl of $\vec{F}$ = Flux$(\nabla \times \vec{F})$

$$\int_{\partial D} \alpha_{\vec{F}} = \int_{D} d\alpha_{\vec{F}}$$

Gauss': Flux of $\vec{F}$

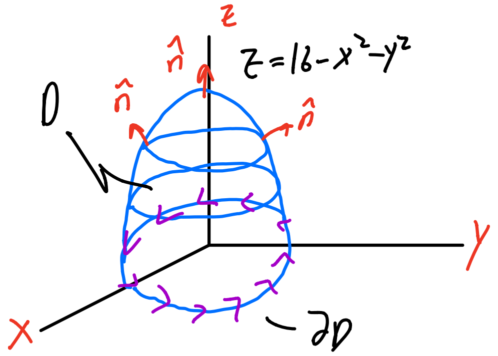
$$\int_{\partial D} \varphi_{\vec{F}} = \int_{D} d\varphi_{\vec{F}} = \int_{D} \nabla \cdot \vec{F} \, dv/dA$$

See the Similarities.

Plot twist: Everything here is just Stokes' thrm in different dimensions! So. Cool.

Ex Stoke's thm states $\int_{\partial D}\alpha_{\vec F} = \int_D d\alpha_{\vec F}$. Verify that the two integrals are indeed equivalent by finding the flux of the curl of $\vec F = y\hat i + 4z\hat j - 6x\hat k$ out of the surface $z = 16-x^2-y^2,\ z \geq 0$. Given: $\int_0^{2\pi} \sin^2 x\,dx = \pi$.

Sketch surface:

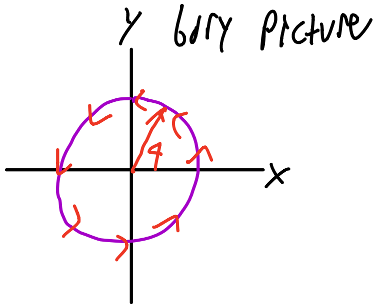


Verify LHS first: $\boxed{\int_{\partial D}\alpha_{\vec F}} = \int_D d\alpha_{\vec F}$. Which will be easier? (LHS v RHS)

$$\alpha_{\vec F} = y\,dx + 4z\,dy - 6x\,dz$$

∂D: $z=0 \to x^2+y^2=16$

bdry D is circle $R=4$ in xy plane



$$\gamma(t) = \begin{bmatrix} 4\cos t \\ 4\sin t \end{bmatrix} \quad 0 \leq t \leq 2\pi$$

$$\gamma'(t) = \begin{bmatrix} -4\sin t \\ 4\cos t \end{bmatrix}$$

$$\Longrightarrow \int_{\partial D}\alpha = \int \Big((4\sin t)(-4\sin t) + 4(0)(4\cos t) - 6(4\cos t)(0)\Big)dt$$

$$= -16\int_0^{2\pi}\sin^2 t\ dt = \boxed{-16\pi}$$

Now try RHS: $\int_D d\alpha$

$$\alpha_{\vec{F}} = y\,dx + 4z\,dy - 6x\,dz$$

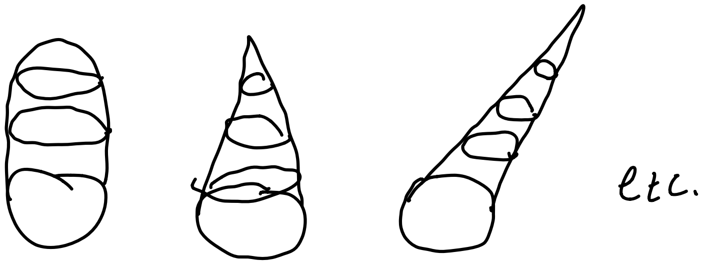
$$d\alpha_{\vec{F}} = dy \wedge dx + 4\,dz \wedge dy - 6\,dx \wedge dz$$

put everything in basis!

$$= -dx \wedge dy - 4\,dy \wedge dz + 6\,dz \wedge dx$$

$$\int_{\partial D} \alpha = \int_{D} d\alpha = \int_{D} -dx \wedge dy - 4\,dy \wedge dz + 6\,dz \wedge dx$$

So what is D? Well, D is the interior to ∂D. Recall ∂D is circle $R=4$ in xy plane. So, the paraboloid does have this as the bdry, but are there any other surfaces that have the circle $R=4$ in xy plane? ... Yes!

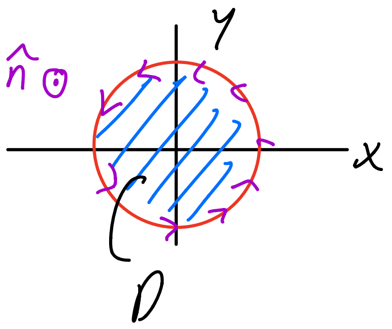

etc.

THE CHOICE IS OURS! Aint that amazing?

Yes. Yes it is.

So whats the easiest choice for D, w/ the one condition being the $\underset{\substack{\uparrow \\ \text{"boundary"}}}{\partial D}$ = circle radius 4 in xy plane?

A disk of R=4 in xy plane



from above...

$$\longrightarrow \int_{\substack{D = \text{Disk} \\ R=4 \\ \text{in xy plane}}} -dx\wedge dy \underbrace{- 4\,\cancel{dy\wedge dz}^{\,0} + 6\,\cancel{dz\wedge dx}^{\,0}}_{\text{projected area onto these planes is 0!!!}}$$

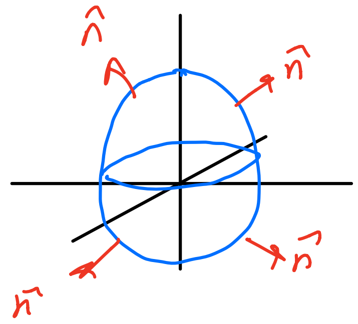
$$-\int_D dx\wedge dy = -\pi(4)^2 = \boxed{-16\pi}$$

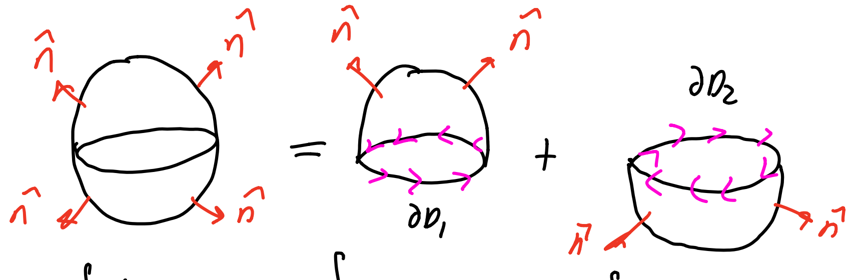
Cool, right? Yes.

Ex | Compute the flux of the curl of $\vec{F} = e^{z\sin(xy)}\,\hat{i} + (z - xy^3)\,\hat{j} + z^3\cos(e^{xy-z})\,\hat{k}$

out of the sphere of radius R centered at the point (2, 3, 1).

$$\int_{\partial D} \alpha = \int_D d\alpha = \text{Flux}(\text{curl}(\vec{F})).$$





$$\int_{\partial D} \alpha = \int_{\partial D_1} \alpha + \int_{\partial D_2} \alpha$$

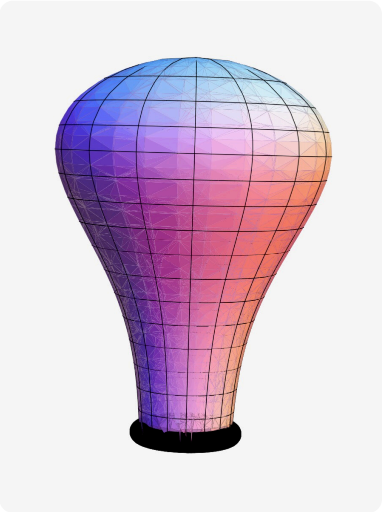
$$= \boxed{0}$$

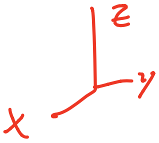
Flux of Curl through any Closed Surface is 0

Ex Suppose you have the lightbulb shaped region as shown below. The bottom bdry of the light bulb is the unit circle $x^2+y^2=1$.

Compute the flux of the curl of

$$\vec{F} = \left(e^{z^2-2z}x,\ \sin(xyz)+y+1,\ e^{z^2}\sin(z^2)\right)$$ out of light bulb.



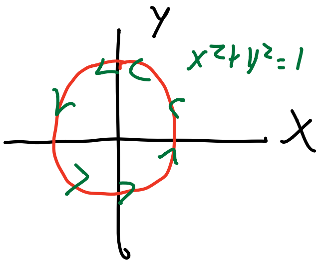


Flux of curl = stokes'!!

$\int_{\partial D}\alpha = \int_{D} d\alpha$; D = surface of bulb

∂D = unit circle in xy plane

LHS waaay easier than RHS:



In xy plane; $z=0$, $dz=0$

$\alpha_{\vec{F}} \to \alpha = x\,dx + (y+1)dy$

$\int_{\partial D} x\,dx + (y+1)dy$; $\gamma(\theta) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ $t: 0 \to 2\pi$.

$\gamma' = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$

$$\int_0^{2\pi}\left(\cos\theta(-\sin\theta) + (\sin\theta+1)(\cos\theta)\right)d\theta$$

$$\int_0^{2\pi}\left(-\sin\theta\cos\theta + \sin\theta\cos\theta + \cos\theta\right)d\theta$$

$$= \int_0^{2\pi}\cos\theta\,d\theta = \sin\theta\Big]_0^{2\pi} = \boxed{0}$$

Ex Compute $\int -xy^2\,dx + y\,dy$ over the circle of radius R in the xy plane, CCW.

$$\alpha = -xy^2\,dx + y\,dy \;;$$

$$\int_{\partial D} \alpha = \int_D d\alpha$$

$\uparrow$ closed loop

try RHS. LHS will be hard w/ $x\,y^2$!!

$$d\alpha = (-y^2\,dx - 2xy\,dy) \wedge dx + dy \wedge dy$$

$$= -2xy\,dy \wedge dx = +2xy\,dx \wedge dy$$

$$\int_{\substack{\text{disk}\\\text{radius}\\R}} 2xy\,dx \wedge dy = 2\iint_{\substack{\text{disk}\\\text{Radius}\\R}} xy\,dA \;; \quad x = r\cos\theta \quad y = r\sin\theta \qquad dA = r\,dr\,d\theta$$

$$= 2\iint r\cos\theta\, r\sin\theta\, r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^{R} \cos\theta\,\sin\theta\; r^3\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{R} \sin(2\theta)\,r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{R^4}{4}\sin(2\theta)\,d\theta$$

$u = 2\theta \to du = 2d\theta$

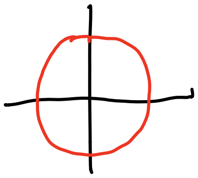
$$= \frac{R^4}{4}\left[\frac{1}{2}\int \sin(u)\,du\right]$$

$$= \frac{R^4}{8}\Big[-\cos(2\theta)\Big]_0^{2\pi} = \frac{R^4}{8}\Big[\cos(2\theta)\Big]_{2\pi}^{0}$$

$$= \frac{R^4}{8}\left[\cos(0) - \cos(4\pi)\right] = \frac{R^4}{8}\left[1-1\right] = \boxed{0}$$

Could we have predicted that?

$$\int_{\substack{\text{Circle}\\ \text{radius}\\ R}} -xy^2\,dx + y\,dy$$



Yes.